SINGULARITIES OF RATIONAL CURVES ON K3 SURFACES

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ABSTRACT. We proved that every rational curve in the primitive class of a general K3 surface is nodal.

1. INTRODUCTION

The purpose of this paper is to prove the following theorem.

Theorem 1.1. *For $g \geq 3$, all rational curves in the linear system $|\mathcal{O}_S(1)|$ on a general primitive K3 surface S in $\mathbb{P}^g$ are nodal.*

The motivation to study this problem has been explained in [C]. Basically, we want to justify the beautiful formula of Yau and Zaslow [Y-Z], which counts the number of rational curves in $|\mathcal{O}_S(1)|$ on a primitive K3 surface $S \subset \mathbb{P}^n$. The primary consequence of Theorem 1.1 is that the formula of Yau and Zaslow actually gives the number of rational curves in $|\mathcal{O}_S(1)|$ on a general K3 surface $S \subset \mathbb{P}^n$.

It has been proved in [C] that Theorem 1.1 is true for $g \leq 9$ and $g = 11$ by degenerating a general K3 surface to a trigonal K3 surface. However, for g large, we have to further degenerate a trigonal K3 surface. The complexities involved in this process prevent us carrying out the proof for any g. Although here we still use a degeneration argument, our approach is entirely different. Here is a rough sketch of the proof.

We start with the degeneration of a K3 surface to the union of two rational surfaces. Let $X \to \Delta$ be a family of K3 surfaces of genus g over the disk Δ whose central fiber $X_0 = R = R_1 \cup R_2$ is the union of two rational surfaces R_1 and R_2 which meet transversely along an elliptic curve $E = R_1 \cap R_2$. We may choose R_i to be $\mathbb{P}^1 \times \mathbb{P}^1$ if g is odd and choose R_i to be $\mathbb{F}_1$ if g is even for $i = 1, 2$. Let us consider the case that $g = 2k + 1$ is odd. We may construct X in such a way that the limit of primitive line bundles $\mathcal{O}_{X_t}(1)$ on X_t is the line bundle $\mathcal{O}_R(1)$ on $X_0 = R$, whose restriction to each $R_i \cong \mathbb{P}^1 \times \mathbb{P}^1$ is the line bundle of type $(1, k)$. So if we have a family $\Upsilon \subset X$ of rational curves

over Δ, whose general fiber Υ_t is a rational curve in the linear series $|\mathcal{O}_{X_t}(1)|$ for each t, the central fiber Υ_0 will be a curve in the linear series $|\mathcal{O}_R(1)|$ and hence $\Upsilon_0 = \Sigma_1 \cup \Sigma_2$ where Σ_i is a curve of type $(1, k)$ on R_i. Our trivial observation is that Υ_t is nodal if Υ_0 is nodal. However, Υ_0 could fail to be nodal where

1. it has a reduced (i.e. isolated) singularity other than a node; or
2. it is nonreduced.

The first case turns out much easier to handle than the second. This is basically due to the fact that each Σ_i is a curve of type $(1, k)$ on $\mathbb{P}^1 \times \mathbb{P}^1$. So all the isolated singularities of Σ_i are nodes. If $\Upsilon_0 = \Sigma_1 \cup \Sigma_2$ has an isolated singularity other than a node, it must be one of the intersections between Σ_1 and Σ_2 on E. The deformation of such singularities has been studied in [C]. With a bit more care, we are able to show that these singularities deform to nodes on the general fiber Υ_t. However, if Υ_t is a rational curve in a multiple of the primitive class, Σ_i might have isolated singularities other than nodes which have to be taken care of. This is one of the major obstacles to generalize Theorem 1.1 to all rational curves on K3 surfaces.

To handle the second case, i.e., to handle the nonreduced components of Υ_0, we first divide them into three types, which we will call Type I, II or III chain (see Sec. 2), respectively. The deformation of Υ_0 along a Type I chain is studied in Sec. 3. The basic technique used there is to normalize the total family along the Type I chain after a suitable base change. The deformation of Υ_0 along a Type II chain is studied in Sec. 4, where we build our argument upon a lower bound estimation on the δ-invariant of Υ_t in the neighborhood of a Type II chain. The deformation of Υ_0 along a Type III chain is studied in Sec. 5. This turns out to be the hardest case among the three. A two-stage degeneration is used, First, we degenerate a general K3 surface to an elliptic K3 (see Sec. 5); and then we degenerate an elliptic K3 to the union of two rational surfaces described above. The degeneration of a K3 surface to an elliptic K3 is also an important step in Bryan and Leung's work [B-L], although the elliptic K3 surfaces they used are different from the ones we use.

As a side note, there have been several progresses made on the enumeration problems on K3 surfaces following Yau and Zaslow's work. A. Beauville pointed out that the numbers Yau and Zaslow obtained are the numbers of rational curves in $|\mathcal{O}_S(1)|$ with each curve counted with certain multiplicity [B], the multiplicity of a rational curve only depends on its singularities and is 1 if the curve is nodal. He gave an algebraic definition of the multiplicy. Later B. Fantechi, L. Göttsche

and D. Straten proved that the multiplicy assigned by Beauville to a rational curve is positive. Recently, J. Bryan and N.C. Leung obtained Yau-Zaslow's formula via a completely different approach [B-L].

Conventions.

1. Throughout the paper, we will work exclusively over $\mathbb{C}$.
2. Here a general K3 surface S in $\mathbb{P}^g$ refers to a general primitive K3 surface in $\mathbb{P}^g$, where the number g is called the genus of S by convention.
3. Since we are working over $\mathbb{C}$, we will use analytic geometry whenever possible. Hence we will use analytic neighborhoods of points instead of Zariski open neighborhoods in most cases, while you may always replace them by formal or etale neighborhoods.

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2. Degeneration of K3 Surfaces and Limiting Rational Curves

2.1. **Degeneration of K3 surfaces.** V. Kulikov classified all the possible degenerations of K3 surfaces in [K]. Here we only need one of the simplest cases: the degeneration of K3 surfaces into a union of two rational scrolls as used in [CLM]. This is also the degeneration used in the proof of the existence of rational curves on K3 surfaces in [C].

Following the notations in [CLM], let $R = R_1 \cup R_2$ be the union of two rational surfaces R_1 and R_2, which meet transversely along a smooth elliptic curve $E = R_1 \cap R_2$. For our purpose, we only need the cases that either $R_1, R_2 \cong \mathbb{P}^1 \times \mathbb{P}^1$ or $R_1, R_2 \cong \mathbb{F}_1$, where $\mathbb{F}_1$ is the rational ruled surface $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-1))$ over $\mathbb{P}^1$.

We may represent such R by the tuple (E, i_1, i_2) where $i_1 : E \to R_1$ and $i_2 : E \to R_2$ are the embeddings of E to R_1 and R_2, respectively. Two unions R and R' represented by (E, i_1, i_2) and (E', i_1', i_2') are isomorphic if and only if $E \cong E'$ and there exist isomorphisms $\varphi : E' \to E$, $\phi_1 : R_1 \to R_1$ and $\phi_2 : R_2 \to R_2$ such that $i_1' = \phi_1 \circ i_1 \circ \varphi$ and $i_2' = \phi_2 \circ i_2 \circ \varphi$. Then it is not hard to see that such R's form an irreducible moduli space of dimension 4.

Let C_i and F_i be two generators of $\text{Pic}(R_i)$ with $C_i \cdot F_i = 0$, $F_i^2 = 0$ and $C_i^2 = 0$ if $R_i \cong \mathbb{P}^1 \times \mathbb{P}^1$; $C_i^2 = -1$ if $R_i \cong \mathbb{F}_1$, for $i = 1, 2$. We

use the notation $\mathcal{O}_R(aC + bF)$ to denote the line bundle on R whose restrictions to R_i are $\mathcal{O}_{R_i}(aC_i + bF_i)$, if such line bundle exists.

It is not hard to see that the dualizing sheaf ω_R of R is trivial and $H^1(R, \mathbb{Z}) = 0$. So it is expected that a general deformation of R, say $X \to \Delta$ with $X_0 = R$, is a family of K3 surfaces. But since a general R is not projective, the general fiber of X is not algebraic. It is easy to see that R is projective if and only if there exists two ample line bundles $L_1 \in \text{Pic } R_1$ and $L_2 \in \text{Pic } R_2$ such that $L_1|_E = L_2|_E$. One obvious choice of L_i is $L_i = \mathcal{O}_{R_i}(C_i + kF_i)$. So we are considering the unions R of the following type.

Let $R = R_1 \cup R_2$ be a union of rational surfaces described as above, which further satisfies $\mathcal{O}_E(C_1 + kF_1) = \mathcal{O}_E(C_2 + kF_2)$ for some $k \geq 1$ if $R_i \cong \mathbb{P}^1 \times \mathbb{P}^1$ and $k \geq 2$ if $R_i \cong \mathbb{F}_1$. We will call such R *a union of scrolls of genus g*, where $g = 2k + 1$ if $R_i \cong \mathbb{P}^1 \times \mathbb{P}^1$ and $g = 2k$ if $R_i \cong \mathbb{F}_1$.

Notice that the relation $\mathcal{O}_E(C_1 + kF_1) = \mathcal{O}_E(C_2 + kF_2)$ imposes one extra condition on the tuple (E, i_1, i_2) which represents R. So the moduli space of unions of scrolls of fixed genus g has dimension 3. With a little extra effort, one can see that the moduli space is irreducible.

We are also interested in a special union of scrolls $R = R_1 \cup R_2$ which satisfies $\mathcal{O}_E(C_1) = \mathcal{O}_E(C_2)$ and $\mathcal{O}_E(F_1) = \mathcal{O}_E(F_2)$. We will call such R *a degenerated (or special) union of scrolls*. It follows from the similar argument as before that degenerated unions of scrolls form an irreducible moduli space of dimension 2.

It was proved in [CLM] that a general union of scrolls R of genus g lies on the boundary of a complete family of K3 surfaces of genus g. The construction is straightforward to carry out by embedding R to $\mathbb{P}^g$ by the complete linear series $|\mathcal{O}_R(C + kF)|$. Then R lies on the component of the Hilbert scheme whose general point is a primitive K3 surfaces in $\mathbb{P}^g$.

However, we need a little bit more. We want to find a complete family of K3 surfaces of genus g whose boundary also contains degenerated unions of scrolls. The previous construction fails since $\mathcal{O}_R(C + kF)$ is ample but not very ample on a degenerated union of scrolls R: the morphism $R \to \mathbb{P}^g$ given by $|\mathcal{O}_R(C + kF)|$ maps R to a double scroll. The remedy to this situation is trivial. Instead of using $|\mathcal{O}_R(C + kF)|$, we embed R to a projective space by the complete linear series $|\mathcal{O}_R(l(C + kF))|$ for some large l. Actually, it is enough to take $l = 2$. Namely, we embed R to $\mathbb{P}^{4g-3}$ by $|\mathcal{O}_R(2C + 2kF)|$. And we can show that R lies on the component of the Hilbert scheme whose general point is a K3 surface of genus g in $\mathbb{P}^{4g-3}$. The argument for

this statement is identical to that in [CLM, Sec. 2.2] and we will only formulate it in the following proposition without the proof.

Proposition 2.1. *Let R be a union of scrolls of genus g which is embedded into $\mathbb{P}^{4g-3}$ by $|\mathcal{O}_R(2C + 2kF)|$, where $k = \lfloor g/2 \rfloor$. Let N_R be the normal bundle of $R \subset \mathbb{P}^{4g-3}$ and $T_R^1 = \mathcal{E}xt^1(\Omega_R, \mathcal{O}_R)$. Then*

1. $H^1(N_R) = 0$;
2. $\dim H^0(N_R) = \dim \mathbb{P}\mathrm{GL}(4g - 2) + 19$;
3. *the natural map from $H^0(N_R)$ to $H^0(T_R^1)$ is surjective;*
4. *R is represented by a smooth point in the component $\mathcal{H}_g$ of the Hilbert scheme whose general point is a K3 surface of genus g in $\mathbb{P}^{4g-3}$.*

Let $\mathcal{R}_g \subset \mathcal{H}_g$ be the locus in $\mathcal{H}_g$ consisting of points representing unions of scrolls and let $\mathcal{R}_g^0 \subset \mathcal{R}_g$ be the locus consisting of points representing degenerated unions of scrolls. Every automorphism of a union of scrolls $R = R_1 \cup R_2$ induces an automorphism of R_1 which maps the double curve $E = R_1 \cap R_2$ to itself. Obviously, there are only finitely many automorphisms of R_1 with this property. So the automorphism group of R is finite. Therefore, $\dim \mathcal{R}_g = \dim \mathcal{R}_g^0 + 1 = \dim \mathbb{P}\mathrm{GL}(4g - 2) + 3$.

Let $\widetilde{\mathcal{H}_g}$ be the blowup of $\mathcal{H}_g$ along the closure of $\mathcal{R}_g$ and let $\widetilde{\mathcal{R}_g}$ and $\widetilde{\mathcal{R}_g^0}$ be the total transforms of $\mathcal{R}_g$ and $\mathcal{R}_g^0$ under the map $\widetilde{\mathcal{H}_g} \to \mathcal{H}_g$, respectively.

For any $[R] \in \mathcal{R}_g$, it is not hard to see that the tangent space $T_{\mathcal{R}_g,[R]}$ of $\mathcal{R}_g$ at $[R]$ lies inside the kernel of the surjection $T_{\mathcal{H}_g,[R]} = H^0(N_R) \to H^0(T_R^1)$. On the other hand, $\dim \mathcal{R}_g = \dim H^0(N_R) - \dim H^0(T_R^1)$. Therefore, $\mathcal{R}_g$ is smooth everywhere. Hence $\widetilde{\mathcal{H}_g}$ is smooth in the neighborhood of $\widetilde{\mathcal{R}_g}$.

Let $\mathcal{S}_g = \{([X], p) : p \in X\} \subset \mathcal{H}_g \times \mathbb{P}^{4g-3}$ be the universal family over $\mathcal{H}_g$ and $\widetilde{\mathcal{S}_g} = \mathcal{S}_g \times_{\mathcal{H}_g} \widetilde{\mathcal{H}_g}$.

It is not hard to see that every point of $\widetilde{\mathcal{R}_g}$ can be uniquely represented by a pair $([R], s)$ with $[R] \in \mathcal{R}_g$ and $s \in \mathbb{P}H^0(T_R^1)$, which we will write as $[R^s]$. Notice that T_R^1 is a sheaf supported along E whose restriction to E is the line bundle $N_{E/R_1} \otimes N_{E/R_2}$, where N_{E/R_1} and N_{E/R_2} are the normal bundle of E in R_1 and R_2, respectively. A geometric interpretation of this blowup process can be put as follows. Let $\pi : \Delta \to \widetilde{\mathcal{H}_g}$ be a morphism from the disk Δ to $\widetilde{\mathcal{H}_g}$, where $\pi(0) = [R^s] \in \widetilde{\mathcal{R}_g}$ and $\pi(\Delta) \not\subset \widetilde{\mathcal{R}_g}$. Then the corresponding family $X = \widetilde{\mathcal{S}_g} \times_{\widetilde{\mathcal{H}_g}} \Delta \to \Delta$ has sixteen rational double points which are the vanishing loci of s.

Remark 2.1. Here we use the term "rational double points" in a broad sense. Let $X \to \Delta$ be a one-parameter family of surfaces over the disk Δ. A point $p \in X$ is called a rational double point of the family X over Δ if X is locally isomorphic to $\mathrm{Spec}\,\mathbb{C}[[x, y, z, t]]/(xy - t^\alpha z)$ at p, where t parameterizes the disk Δ and α is a positive integer. If $\alpha = 1$, p is a rational double point of the threefold X in the usual sense, which can be resolved by blowing up X at p and then blowing down along one of the rulings of the exceptional divisor $\mathbb{P}^1 \times \mathbb{P}^1$. If $\alpha > 1$, we may resolve p in the same way as in [G-H, Appendix C, p. 39]. But we will take a more direct approach here by choosing an neighborhood U of p and simply letting $\widetilde{U} \subset U \times \mathbb{P}^1$ be defined by

$$(2.1) \qquad \frac{x}{t^\alpha} = \frac{z}{y} = \frac{Z_0}{Z_1}$$

where (Z_0, Z_1) are the homogeneous coordinates of $\mathbb{P}^1$. It is trivial to glue $\widetilde{U}$ with $X \backslash \{p\}$ and arrive at a "resolution" $\widetilde{X}$ of X (rigorously, $\widetilde{X}$ is not a desingularization of X since $\widetilde{X}$ is still singular in dimension one; but now $\widetilde{X}$ is isomorphic to $\mathrm{Spec}\,\mathbb{C}[[x, y, z, t]]/(xy - t^\alpha)$ everywhere along its singular locus, which is all we need). Of course, we may switch x and y in (2.1) and arrive at another resolution of X. Obviously, this corresponds to the "flip" phenomenon of the resolutions of an ordinary three-fold double point.

Let $\mathcal{W}_g$ be the incidence correspondence

$$\mathcal{W}_g = \big\{ ([S], [C]) : [S] \in \widetilde{\mathcal{H}}_g \text{ general},$$
$$\qquad\qquad C \text{ is a rational curve in the primitive class of } S \big\}$$
$$\qquad \subset \widetilde{\mathcal{H}}_g \times |\mathcal{O}_{\mathbb{P}^{4g-3}}(1)|,$$

where we construct $\mathcal{W}_g$ as a subvariety of $\widetilde{\mathcal{H}}_g \times |\mathcal{O}_{\mathbb{P}^{4g-3}}(1)|$ by identifying $[C]$ with $2C \in |\mathcal{O}_S(2)| = |\mathcal{O}_{\mathbb{P}^{4g-3}}(1)|$ (let $\mathcal{O}_S(1)$ be the primitive line bundle on S). Let $\widetilde{\mathcal{W}}_g$ be the closure of $\mathcal{W}_g$ in $\widetilde{\mathcal{H}}_g \times |\mathcal{O}_{\mathbb{P}^{4g-3}}(1)|$.

Obviously, Theorem 1.1 is equivalent to the following statement: for every point $([S], [C])$ on the fiber of $\widetilde{\mathcal{W}}_g \to \widetilde{\mathcal{H}}_g$ over a general point $[S] \in \widetilde{\mathcal{H}}_g$, C is a nodal curve. Obviously, it suffices to verify this statement for every irreducible component of $\widetilde{\mathcal{W}}_g$ which dominates $\widetilde{\mathcal{H}}_g$. So, from now on, we will pretend that $\widetilde{\mathcal{W}}_g$ is irreducible and dominates $\widetilde{\mathcal{H}}_g$.

It is obvious that the map $\widetilde{\mathcal{W}}_g \to \widetilde{\mathcal{H}}_g$ is generically finite. One important step to prove Theorem 1.1 is to figure out what kind of curves $[C]$ lie on the fiber $(\widetilde{\mathcal{W}}_g)_{[R^s]}$ of $\widetilde{\mathcal{W}}_g \to \widetilde{\mathcal{H}}_g$ over a point $[R^s] \in \widetilde{\mathcal{R}}_g$. Of course, $C \in |\mathcal{O}_R(C + kF)|$. Besides, C must be the limit of rational

curves on K3 surfaces. More precisely, $([R^s], [C]) \in (\widetilde{\mathcal{W}}_g)_{[R^s]}$ if and only if there exist a family X of K3 surfaces over Δ and a family $\Upsilon \subset X$ of rational curves where $X_0 = R$, X has sixteen rational double points which are the vanishing loci of s, $\Upsilon_0 = C$ and Υ_t is a rational curve in the primitive class of X_t. It turns out that there are only finitely many curves C in $|\mathcal{O}_R(C + kF)|$ with this property.

Definition 2.1. Let R be a union of scrolls and $s \in \mathbb{P}H^0(T_R^1)$. A curve $C \in |\mathcal{O}_R(C + kF)|$ with $([R^s], [C]) \in \widetilde{\mathcal{W}}_g$ is called a *limiting rational curve on R^s*. When there is no possibility for confusion, we will abbreviate R^s to R and simply call C a limiting rational curve on R.

2.2. Classifications of limiting rational curves.
Let $\pi : \Delta \to \widetilde{\mathcal{H}}_g$ be a morphism from the disk Δ to $\widetilde{\mathcal{H}}_g$ where $\pi(0) = [R^s] \in \widetilde{\mathcal{R}}_g$ and $\pi(t)$ represents a general K3 surface of genus g for $t \in \Delta$ general.

Let $X = \widetilde{\mathcal{S}}_g \times_{\widetilde{\mathcal{H}}_g} \Delta \to \Delta$ be the one parameter family of K3 surfaces corresponding to π. Obviously, X has exactly 16 rational double points $p_1, p_2, ..., p_{16}$ lying on the double curve $E = R_1 \cap R_2$, where $\{p_1, p_2, ..., p_{16}\}$ is the vanishing locus of $s \in \mathbb{P}H^0(T_R^1)$. For a general choice of s, $p_1, p_2, ..., p_{16}$ satisfy the only relation

$$(2.2) \qquad \mathcal{O}_E(p_1 + p_2 + ... + p_{16}) = N_{E/R_1} \otimes N_{E/R_2}$$

where $N_{E/R_i} = \mathcal{O}_E(2C_i + 2F_i)$ if $R_i \cong \mathbb{P}^1 \times \mathbb{P}^1$ and $N_{E/R_i} = \mathcal{O}_E(2C_i + 3F_i)$ if $R_i \cong \mathbb{F}^1$.

Let $\Upsilon \subset X$ be a family of curves over Δ whose general fiber Υ_t is a rational curve in the primitive class of X_t for each t. Let $\widetilde{\Upsilon}$ be the nodal reduction of Υ. Namely, $\widetilde{\Upsilon}$ is a family of stable maps to X such that $\widetilde{\Upsilon} \to X$ factors through Υ and the general fiber $\widetilde{\Upsilon}_t$ of $\widetilde{\Upsilon}$ is the normalization of the general fiber Υ_t of Υ. Notice that such Υ and $\widetilde{\Upsilon}$ exist after a base change.

One of the key lemmas we use to classify limiting rational curves on X_0 is Lemma 2.2 in [C]. We need a slightly stronger version, which is formulated and proved as follows.

Lemma 2.1. *Let $X \subset \Delta^n_{x_1 x_2 ... x_n} \times \Delta_t$ ($n \geq 3$) be the hypersurface cut out by $x_1 x_2 = t^\alpha$ for some $\alpha > 0$, where $\Delta^n_{x_1 x_2 ... x_n}$ is the n-dimensional polydisk parameterized by $(x_1, x_2, ..., x_n)$ and Δ_t is the disk parameterized by t. Let X_0 be the central fiber of X over Δ_t, let $X_0 = R_1 \cup R_2$ where $R_1 = \{x_1 = t = 0\}$ and $R_2 = \{x_2 = t = 0\}$ and let $E = R_1 \cap R_2$. Let S be a flat family of analytic curves over Δ_t and $\pi : S \to X$ be a proper morphism preserving the base Δ_t. Suppose that the image $\pi(S_0)$ of the central fiber S_0 of S meets E properly along a 0-dimensional*

scheme supported at the origin. Let $S_0 = \Gamma_1 \cup \Gamma_2$ with $\pi(\Gamma_1) \subset R_1$ and $\pi(\Gamma_2) \subset R_2$. Then $\pi(\Gamma_1) \cdot R_2 = \pi(\Gamma_2) \cdot R_1$ holds on $\{t = 0\} \cong \Delta^n_{x_1 x_2 \ldots x_n}$.

Proof. Without the loss of generality, we may assume that S is irreducible and smooth; otherwise, we apply the following argument to each irreducible component of its desingularization. Let $\eta : X \to X' = \{x_1 x_2 = t\} \subset \Delta^n_{x_1 x_2 \ldots x_n} \times \Delta_t$ be the morphism sending $(x_1, x_2, \ldots, x_n, t)$ to $(x_1, x_2, \ldots, x_n, t^\alpha)$ and let $p = \eta \circ \pi$. Obviously, it suffices to show that $p_*(\Gamma_1) \cdot R_2 = p_*(\Gamma_2) \cdot R_1$ on X'.

Since $\Gamma_1 \cdot p^*(R_1 + R_2) = 0$ and $(\Gamma_1 + \Gamma_2) \cdot p^*(R_1) = 0$ on S,

$$\Gamma_1 \cdot p^*(R_2) = \Gamma_2 \cdot p^*(R_1).$$

Therefore, $p_*(\Gamma_1) \cdot R_2 = p_*(\Gamma_2) \cdot R_1$ by the projection formula. $\qquad\square$

The following proposition deals with the case that X has a rational double point at the origin, which can be viewed as a corollary of Lemma 2.1.

Corollary 2.1. *Let X, R_1, R_2, E, π, S and S_0 be defined as in Lemma 2.1 except that X is cut out by $x_1 x_2 = t^\alpha x_3$ for some $\alpha > 0$. Suppose that $\pi(S_0)$ contains an irreducible component $\Gamma_1 \subset R_1$ such that the tangent cones of Γ_1 and E at the origin p do not meet properly in the tangent space of R_1 at p. Then $\pi(S_0)$ also contains a component $\Gamma_2 \subset R_2$.*

Proof. We may resolve the double point of X as in Remark 2.1. Let $\widetilde{X} \subset X \times \mathbb{P}^1$ be given by

$$\frac{x_1}{t^\alpha} = \frac{x_3}{x_2} = \frac{Z_0}{Z_1}$$

where (Z_0, Z_1) are the homogeneous coordinates of $\mathbb{P}^1$. Let $\widetilde{S} = S \times_X \widetilde{X}$. Obviously, $\widetilde{X}_0 = \widetilde{R_1} \cup R_2$, where $\widetilde{R_1}$ is the blowup of R_1 along the subscheme $\{x_2 = x_3 = 0\}$. Obviously, $\widetilde{R_1}$ and R_2 meet along $\widetilde{E} = \{x_1 = x_2 = z = t = 0\}$ (let $z = Z_1/Z_0$) and $\widetilde{X}$ is locally defined by

(2.3) $x_1 z = t^\alpha$ and $x_2 = z x_3$

in the neighborhood of $\widetilde{E}$. Let $\widetilde{p} \in \widetilde{E}$ be the point $\{x_1 = x_2 = \ldots = x_n = z = t = 0\}$ and let $\widetilde{\Gamma_1} \subset \widetilde{S}_0$ be the proper transform of Γ_1 under the map $\widetilde{R_1} \to R_1$. The assumption that the tangent cones of Γ_1 and E at p do not meet properly implies that their proper transforms $\widetilde{\Gamma_1}$ and $\widetilde{E}$ under the blowup $\widetilde{R_1} \to R_1$ still meet at a point on the exception divisor, i.e., they meet at $\widetilde{p}$. Now by (2.3), $\widetilde{X}$ is locally defined by $x_1 z = t^\alpha$ in $\Delta^n_{x_1 z x_3 \ldots x_n} \times \Delta_t$ at $\widetilde{p}$. And $\widetilde{\pi}(\widetilde{S}_0)$ has a component $\widetilde{\Gamma_1}$

passing through $\widetilde{p}$ and lying on $\widetilde{R_1}$, where $\widetilde{\pi} : \widetilde{S} \to \widetilde{X}$ is the map induced by π. So by Lemma 2.1, $\widetilde{\pi}(\widetilde{S}_0)$ contains a component $\Gamma_2 \subset R_2$, i.e., $\pi(S_0)$ contains the component $\Gamma_2 \subset R_2$. $\square$

Let $\Upsilon_0 = \Sigma_1 \cup \Sigma_2$ with $\Sigma_i \subset R_i$ for $i = 1, 2$. And let $\Gamma_i \subset \Sigma_i$ be the irreducible component of Σ_i in $|C_i + k_i F_i|$ for some $k_i \le k$ $(i = 1, 2)$.

Let r_{ij} be all the points on E satisfying $\mathcal{O}_E(2r_{ij}) = \mathcal{O}_E(F_i)$ for $i = 1, 2$ and $j = 1, 2, 3, 4$. Notice that if R is a degenerated union of scroll, we have $\{r_{1j}\} = \{r_{2j}\}$. So we will simply let $r_{1j} = r_{2j}$ for $j = 1, 2, 3, 4$ if $[R] \in \mathcal{R}_g^0$.

For two points p and q on E satisfying $\mathcal{O}_E(p + q) = \mathcal{O}_E(F_1)$ or $\mathcal{O}_E(F_2)$, we use the notation $\overline{pq}$ to denote the curve in $|F_1|$ or $|F_2|$ passing through p and q. If R is general in $\mathcal{R}_g$, there is no ambiguity; otherwise, we use $\overline{pq}^{(1)}$ and $\overline{pq}^{(2)}$ to distinguish on which of R_1 and R_2 this curve lies. And we use $\overline{r_{ij}}$ to denote the curve in $|F_i|$ passing through r_{ij}.

Since there are exactly two components on $\widetilde{\Upsilon}_0$ dominates Γ_1 and Γ_2, respectively, we will continue to use Γ_1 and Γ_2 to denote these two components on $\widetilde{\Upsilon}_0$.

Definition 2.2. Let C be a stable curve. The "dual graph" G of C is a graph constructed by representing each component A of C by the vertex $[A]$ and drawing an edge between two vertices $[A]$ and $[B]$ if the corresponding curves A and B meet at a point. We allow multiple edges between $[A]$ and $[B]$ if they meet at more than one point; and if A has a node, we will draw a loop around $[A]$. Let $\deg([A])$ denote the degree of the vertex $[A]$ in G.

A sequence of components $C_1, C_2, ..., C_n$ of C form a chain in G if there is an edge between $[C_i]$ and $[C_{i+1}]$ for $i = 1, 2, ..., n - 1$.

First, we will show that there are only finitely many limiting rational curves on X_0. Namely, there are only finitely many possible configurations for Υ_0.

Proposition 2.2. *Let $[R^s] \in \widetilde{\mathcal{R}_g^0}$ with $s \in \mathbb{P}H^0(T_R^1)$ general. Then there are only finitely many limiting rational curves on R^s. Namely, the fiberation of $\widetilde{\mathcal{W}_g} \to \widetilde{\mathcal{H}_g}$ is finite over $[R^s]$.*

Proof. Since $[R^s] \in \widetilde{\mathcal{R}_g^0}$, $\mathcal{O}_E(C_1) = \mathcal{O}_E(C_2) = \mathcal{O}_E(C)$ and $\mathcal{O}_E(F_1) = \mathcal{O}_E(F_2) = \mathcal{O}_E(F)$. It is not hard to see that if Σ_1 contains a component $\overline{pq}^{(1)}$, Σ_2 must contain $\overline{pq}^{(2)}$ with the same multiplicity and vice versa. Therefore, we necessarily have $\Gamma_1 \cap E = \Gamma_2 \cap E$.

Let q_i be the point on E such that $\mathcal{O}_E(p_i + q_i) = \mathcal{O}_E(F)$ for $i = 1, 2, ..., 16$. Let $r_j = r_{1j} = r_{2j}$ for $j = 1, 2, 3, 4$.

By (2.2), the only relation among $p_1, p_2, ..., p_{16}$ is $\mathcal{O}_E(p_1 + p_2 + ... + p_{16}) = \mathcal{O}_E(4C + 4F)$ or $\mathcal{O}_E(4C + 6F)$. Therefore, the subgroup of $\text{Pic}(E)$ generated by p_l, q_l and r_j is

$$\overset{15}{\underset{l=1}{\oplus}} \mathbb{Z}p_l \oplus \mathbb{Z}r_1 \oplus \mathbb{Z}(4C) \oplus (\mathbb{Z}_2 \oplus \mathbb{Z}_2)$$

where $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ is the subgroup consisting of $r_j - r_1$ for $j = 1, 2, 3, 4$. This group clearly does not contain the divisor C when $p_1, p_2, ..., p_{15}$ and E are general. Therefore, Γ_i must meet E at (at least) one point other than p_l, q_l and r_j for $l = 1, 2, ..., 16$ and $j = 1, 2, 3, 4$.

Suppose that $p \in \Gamma_i \cap E$ and $p \notin \{p_1, p_2, ..., p_{16}, q_1, q_2, ..., q_{16}\}$. Let q be the point on E such that $\mathcal{O}_E(p + q) = \mathcal{O}_E(F)$. We claim that on $\widetilde{\Upsilon}_0$, the curves Γ_1 and Γ_2 are joined by a chain of curves which either contract to one of the two points p and q or dominate one of the two curves $\overline{pq}^{(1)}$ and $\overline{pq}^{(2)}$.

Let $D_1, D_2, ..., D_\gamma$ be the components of $\widetilde{\Upsilon}_0$ which either contract to one of p and q or dominate one of $\overline{pq}^{(i)}$. Let G be the dual graph of $\Gamma_1 \cup \Gamma_2 \cup D_1 \cup D_2 \cup ... \cup D_\gamma$ with each curve A being represented by the vertex $[A]$.

If $p \notin \{r_1, r_2, r_3, r_4\}$, then $\deg([\Gamma_i]) \geq 1$ and $\deg([D_j]) \geq 2$ for $i = 1, 2$ and $j = 1, 2, ..., \gamma$ by Lemma 2.1. So G either contains a cycle, which is impossible, or G is connected, which implies our claim.

If $p = q = r_j$ for some j, then Γ_i must meet E transversely at r_j; otherwise, Γ_i will meet $\overline{r_{ij}}$ at r_j with multiplicity at least 2, which is impossible. Locally at r_j, Υ_0 consists of $\Gamma_1 \cup \Gamma_2$ and a multiple of $\overline{r_{1j}} \cup \overline{r_{2j}}$. Notice that the local intersection number between Γ_i and E at r_j is 1 (odd) and the local intersection number between $\overline{r_{ij}}$ and E at r_j is 2 (even). Therefore, $[\Gamma_1]$ and $[\Gamma_2]$ must lie on the same connected component of G by Lemma 2.1, which also implies our claim.

In conclusion, Γ_i meet E at (at least) one point other than p_l, q_l and r_j. If Γ_i meet E at p and $p \notin \{p_1, p_2, ..., p_{16}, q_1, q_2, ..., q_{16}\}$, then Γ_1 and Γ_2 are joined by a chain of curves on $\widetilde{\Upsilon}_0$ whose image lie in $\overline{pq}^{(1)} \cup \overline{pq}^{(2)}$, where $q \in E$ is the point satisfying that $\mathcal{O}_E(p+q) = \mathcal{O}_E(F)$. So it is not hard to see that Γ_i cannot meet E at more than one point other than p_l and q_l; otherwise, Γ_1 and Γ_2 will be joined by two different chains of curves. In conclusion, Γ_i meets E at exactly one point p other than p_l and q_l for $l = 1, 2, ..., 16$ and $p \neq r_j$ for $j = 1, 2, 3, 4$. Obviously, there are at most finitely many curves in $|C_i + k_i F_i|$ ($k_i \leq k$) with this property.

It remains to show that there are only finitely many possible configurations for a component $\overline{pq}^{(i)} \subset \Sigma_i$. Actually, we claim that if $\overline{pq}^{(i)} \subset \Sigma_i$, then

1. $p, q \in \{p_1, p_2, ..., p_{16}, q_1, q_2, ..., q_{16}\}$; OR
2. $p = q \in \{r_1, r_2, r_3, r_4\}$; OR
3. p or q lies on Γ_i.

Suppose that $\overline{pq}^{(i)} \subset \Sigma_i$, $p, q \notin \{p_1, p_2, ..., p_{16}, q_1, q_2, ..., q_{16}, r_1, ..., r_4\}$ and $p, q \notin \Gamma_i$. Let $D_1, D_2, ..., D_\gamma$ be the components of $\widetilde{\Upsilon}_0$ which either contract to one of p and q or dominate one of $\overline{pq}^{(i)}$. Let G be the dual graph of $D_1 \cup D_2 \cup ... \cup D_\gamma$ with each curve A represented by the vertex $[A]$. Since $p \neq q$, $p, q \notin \{p_1, p_2, ..., p_{16}\}$ and Γ_i does not pass through p and q, $\deg([D_j]) \geq 2$ for $j = 1, 2, ..., \gamma$ by Lemma 2.1. So G must contain a cycle, which is a contradiction.

In summary, there are at most finitely many possible configurations for $\Upsilon_0 = \Sigma_1 \cup \Sigma_2$. So $\widetilde{\mathcal{W}}_g$ is finite over $[R^s] \in \widetilde{\mathcal{R}}_g^0$. $\qquad \square$

Remark 2.2. Since $\widetilde{\mathcal{R}}_g^0 \subset \widetilde{\mathcal{R}}_g$, Proposition 2.2 implies that the fiberation of $\widetilde{\mathcal{W}}_g \rightarrow \widetilde{\mathcal{H}}_g$ is finite over a general point $[R^s] \in \widetilde{\mathcal{R}}_g$.

In the rest of this section, we will assume that the choice of R^s in the construction of X is general in $\widetilde{\mathcal{R}}_g$.

Definition 2.3. For each point $p \in \Sigma_i \cap E$ $(i = 1, 2)$, we use $m_i(p)$ to denote the local intersection multiplicity between Γ_i and E at p (let $m_i(p) = 0$ if Γ_i does not pass through p).

For each $\overline{pq} \subset \Upsilon_0$, let $\mu(\overline{pq})$ denote the multiplicity of $\overline{pq}$ in Υ_0.

Definition 2.4. An F-chain on R is a union of m distinct curves $C = \overline{q_0 q_1} \cup \overline{q_1 q_2} ... \cup \overline{q_{m-1} q_m}$, where, as the notation suggests, $\overline{q_l q_{l+1}}$ is a curve in either $|F_1|$ or $|F_2|$. We call m the length of C. If $m = 0$, we let C be the point q_0.

A maximal F-chain C in Υ_0 is an F-chain $C \subset \Upsilon_0$ and it is maximal in the sense that there does not exist an F-chain $C' \subset \Upsilon_0$ containing C as a proper subset.

Since we will always deal with maximal F-chains in Υ_0, we will simply call them F-chains in Υ_0.

We divide the F-chains in Υ_0 into three types.

1. A chain C is called a *Type I* chain if $p_l \notin C$ for $l = 1, 2, ..., 16$ and $r_{ij} \notin C$ for $i = 1, 2$ and $j = 1, 2, 3, 4$.
2. A chain C is called a *Type II* chain if $p_l \in C$ for some $1 \leq l \leq 16$.
3. A chain C is called a *Type III* chain if $r_{ij} \in C$ for some $1 \leq i \leq 2$ and $1 \leq j \leq 4$.

Notice that a Type I or II chain could consist of a single point.

Proposition 2.3. *There is exactly one Type I chain in Υ_0, whose length is even. Let $\overline{q_0q_1} \cup \overline{q_1q_2} \cup ... \cup \overline{q_{2l-1}q_{2l}} \subset \Upsilon_0$ be the Type I chain and assume that $\overline{q_0q_1} \subset R_1$ without the loss of generality. Then*

$$\mu(\overline{q_0q_1}) = \mu(\overline{q_1q_2}) = ... = \mu(\overline{q_{2l-1}q_{2l}}) = \beta = m_2(q_0) = m_1(q_{2l}) = \beta \ and$$
$$m_1(q_0) = m_1(q_1) = ... = m_1(q_{2l-1})$$
$$= m_2(q_1) = m_2(q_2) = ... = m_2(q_{2l}) = 0$$

for some $\beta > 0$. There are exactly $2l$ components $D_1, D_2, ..., D_{2l}$ on $\widetilde{\Upsilon}_0$ where each D_i dominates $\overline{q_{i-1}q_i}$ with a degree β map totally ramified at q_{i-1} and q_i. The components $\Gamma_2, D_1, D_2, ..., D_{2l}, \Gamma_1$ form a chain in the dual graph of $\widetilde{\Upsilon}_0$.

Proof. Obviously, every point in $C \cap E$ for a Type II or Type III chain C, as an element in the Picard group $\text{Pic}(E)$, lies in the subgroup generated by p_l ($l = 1, 2, ..., 16$) and r_{ij} ($i = 1, 2$ and $j = 1, 2, 3, 4$), which, by (2.2), is

$$\left(\bigoplus_{l=1}^{15} \mathbb{Z}p_l \right) \oplus \left(\bigoplus_{i=1}^{2} \mathbb{Z}r_{i1} \right) \oplus \mathbb{Z}(2C_1 + 2C_2) \oplus (\mathbb{Z}_2 \oplus \mathbb{Z}_2)^{\oplus 2}$$

where two copies of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ are the subgroups consisting of $r_{ij} - r_{i1}$ ($j = 1, 2, 3, 4$) for $i = 1, 2$, respectively. Obviously, $C_1 + kF_1 = C_2 + kF_2$ does not lie in this group, i.e., it is not generated by p_l and r_{ij}, for R and s general. Therefore there is at least one Type I chain.

Let $\overline{q_0q_1} \cup \overline{q_1q_2} \cup ... \cup \overline{q_{m-1}q_m}$ form a Type I chain in Υ_0. It is not hard to see that $q_0, q_m \in \Gamma_1 \cup \Gamma_2$.

We let $D_1, D_2, ..., D_\gamma$ be the components on $\widetilde{\Upsilon}_0$ dominating the curves $\overline{q_0q_1}, \overline{q_1q_2}, ..., \overline{q_{m-1}q_m}$ and $H_1, H_2, ..., H_\mu$ be the components contracting to the points $q_0, q_1, q_2, ...q_m$.

Let G be the dual graph of $\Gamma_1, D_1, D_2, ..., D_\gamma, \Gamma_2, H_1, H_2, ..., H_\mu$ with each curve A being represented by the vertex $[A]$. Obviously, G contains no circuit and is consequently a "forest" (a disjoint union of trees). By Lemma 2.1, $\deg([D_i]) \geq 2$ for $i = 1, 2, ..., \gamma$ and $\deg([\Gamma_1]) + \deg([\Gamma_2]) \geq 2$. And since $\deg([H_j]) \geq 3$, G has at least $\gamma + 1 + 3\mu/2$ edges with $\gamma + 2 + \mu$ vertices. On the other hand, G has at most $\gamma + 1 + \mu$ edges since it is a forest. So we must have

1. $\mu = 0$, i.e., there are no components of $\widetilde{\Upsilon}_0$ contracting to the points $q_0, q_1, q_2, ..., q_m$;
2. G is a tree;
3. $\deg([D_i]) = 2$ for $i = 1, 2, ..., \gamma$;
4. $\deg([\Gamma_1]) = \deg([\Gamma_2]) = 1$.

And since we assume that $\overline{q_0q_1} \subset R_1$, we must have $q_0 \in \Gamma_2$ and $q_m \in \Gamma_1$. Hence m must be even, say $m = 2l$.

The picture of G is very clear now. The vertices of G form a chain after some ordering. Without the loss of generality, we may assume that $[\Gamma_2], [D_1], [D_2], ..., [D_\gamma], [\Gamma_1]$ form a chain in that order. Obviously, D_1 must dominate $\overline{q_0 q_1}$ and there is no other D_i $(i \neq 1)$ dominating $\overline{q_0 q_1}$; otherwise, $\deg([\Gamma_2]) \geq 2$ by Lemma 2.1. Similarly, D_2 must dominate $\overline{q_1 q_2}$ and there is no other D_i dominating $\overline{q_1 q_2}$; otherwise, $\deg([D_1]) \geq 3$ by Lemma 2.1. This line of argument goes on and finally shows that $\gamma = 2l$ and each D_i dominates $\overline{q_{i-1} q_i}$ for $i = 1, 2, ..., 2l$. Also, the map from D_i to $\overline{q_{i-1} q_i}$ must be totally ramified at q_{i-1} and q_i; otherwise, $\deg([D_i]) \geq 3$ by Lemma 2.1. Since D_i is rational, the map from D_i to $\overline{q_{i-1} q_i}$ is only ramified at q_{i-1} and q_i.

Let $\beta = m_2(q_0)$. Obviously, Γ_2 does not pass through $q_1, q_2, ..., q_{2l}$ and Γ_1 does not pass through $q_0, q_1, ..., q_{2l-1}$; otherwise, either $\deg([\Gamma_2]) \geq 2$ or $\deg([\Gamma_1]) \geq 2$. So $m_2(q_1) = m_2(q_2) = ... = m_2(q_{2l}) = m_1(q_0) = m_1(q_1) = ... = m_1(q_{2l-1}) = 0$. Hence each $\overline{q_{i-1} q_i}$ have multiplicity exactly β in Υ_0 for $i = 1, 2, ..., 2l$. Therefore, the map from D_i to $\overline{q_{i-1} q_i}$ has degree β. This also implies that $m_1(q_{2l}) = \beta$.

Hence Γ_1 and Γ_2 are joined by a chain of curves $D_1 \cup D_2 \cup ... \cup D_{2l}$ on $\widetilde{\Upsilon}_0$ whose images are contained in $\overline{q_0 q_1} \cup \overline{q_1 q_2} \cup ... \cup \overline{q_{2l-1} q_{2l}}$. Therefore, there is only one Type I chain in Υ_0; otherwise, Γ_1 and Γ_2 will be joined by two different chains of curves on $\widetilde{\Upsilon}_0$. $\qquad \square$

To study the behavior of a family Υ of curves near a subscheme $S \subset \Upsilon_0$, it is usually very convenient to take an analytic neighborhood U of S in Υ and study U instead of Υ (or alternatively, study the formal completion of Υ along S). Notice that even if Υ is irreducible, U could be reducible after a base change. We call a component of U which is irreducible under any base changes "a locally irreducible component of Υ around S". And if U is irreducible under any base changes, we call Υ is locally irreducible around S.

Here is a trivial remark. Let $S \subset \Upsilon_0$ be a closed subscheme of Υ_0 and $\widetilde{S}$ be the total transform of S under the map $\widetilde{\Upsilon} \to \Upsilon$. Then Υ is locally irreducible around S if and only if $\widetilde{S}$ is connected.

Let us write a Type II chain in the form $\cup_{i=-l}^{m-1} \overline{q_i q_{i+1}}$, where $q_0 \in \{p_1, p_2, ..., p_{16}\}$.

Proposition 2.4. Let $\cup_{i=-l}^{m-1} \overline{q_i q_{i+1}}$ be a Type II chain in Υ_0 with $q_0 \in \{p_1, p_2, ..., p_{16}\}$. Without the loss of generality, assume that $q_0 = p_1$ and $\overline{q_0 q_1} \subset R_1$.

Let Y be a locally irreducible component of Υ around $\cup_{i=-l}^{m-1} \overline{q_i q_{i+1}}$. Then

1. $\mu(\overline{q_{-l} q_{-l+1}}) \leq \mu(\overline{q_{-l+1} q_{-l+2}}) \leq ... \leq \mu(\overline{q_{-1} q_0})$ and $\mu(\overline{q_0 q_1}) \geq \mu(\overline{q_1 q_2}) \geq ... \geq \mu(\overline{q_{m-1} q_m})$;

2. $m_1(q_0), m_2(q_0) \leq 1$;
3. *for each q_j $(j \neq 0)$, either $m_1(q_j) = 0$ or $m_2(q_j) = 0$;*
4. Y_0 *is one of the following*
   (a) *$Y_0 = \Delta(w)$ where w is the intersection between $\Gamma_1 \cup \Gamma_2$ and $\overline{q_i q_{i+1}}$ other than the points q_i and q_{i+1} for some i and $\Delta(w) \subset \Gamma_1 \cup \Gamma_2$ is a disk centered at w regarded as an analytic neighborhood of w on Γ_1 or Γ_2;*
   (b) *$Y_0 = \Delta(q_0)$ where $\Delta(q_0) \subset \Gamma_1 \cup \Gamma_2$ is a disk centered at q_0 regarded as an analytic neighborhood of q_0 on Γ_1 or Γ_2;*
   (c) *$(Y_0)_{\mathrm{red}} = \cup_{i=0}^{n-1} \overline{q_i q_{i+1}} \cup \Delta(q_n)$ for some $n > 0$, where $\Delta(q_n)$ is a disk centered at q_n regarded as an analytic neighborhood of q_n on Γ_1 or Γ_2, $\Delta(q_n)$ and $\overline{q_{n-1} q_n}$ lie on the different R_j's and $\overline{q_i q_{i+1}}$ has multiplicity $E \cdot \Delta(q_n)$ in Y_0;*
   (d) *$(Y_0)_{\mathrm{red}} = \cup_{i=-n}^{-1} \overline{q_i q_{i+1}} \cup \Delta(q_{-n})$ for some $n > 0$, where $\Delta(q_{-n})$ is a disk centered at q_{-n} regarded as an analytic neighborhood of q_{-n} on Γ_1 or Γ_2, $\Delta(q_{-n})$ and $\overline{q_{-n} q_{-n+1}}$ lie on the different R_j's and $\overline{q_i q_{i+1}}$ has multiplicity $E \cdot \Delta(q_{-n})$ in Y_0.*

Remark 2.3. The statements in Proposition 2.4 may need some further explanation. Let U be an analytic neighborhood of $\cup_{i=-l}^{m-1} \overline{q_i q_{i+1}}$ in Υ. Then the central fiber U_0 of U consists of $\cup_{i=-l}^{m-1} \overline{q_i q_{i+1}}$ plus a few disks which are "pieces" of Γ_1 and Γ_2. If Γ_i meets $\overline{q_j q_{j+1}}$ at $w \neq q_j, q_{j+1}$, there is a disk $\Delta(w) \subset \Gamma_i$ centered at w on U_0; if Γ_i passes through q_j, there is a disk $\Delta(q_j) \subset \Gamma_i$ centered at q_j on U_0. Let $\widetilde{U} = U \times_\Upsilon \widetilde{\Upsilon}$ be the nodal reduction of U. Each locally irreducible component Y of U corresponds to a connected component of the dual graph of $\widetilde{U}_0$ and vice versa.

First, locally at $w = \Gamma_i \cap \overline{q_j q_{j+1}}$ with $w \neq q_j, q_{j+1}$, our statements about Y_0 show that Γ_i $(\Gamma_i \supset \Delta(w))$ is not joined to any component dominating $\overline{q_j q_{j+1}}$ by a chain of curves contracting to w on $\widetilde{\Upsilon}_0$. In a more intuitive language, Γ_i is "separated" from $\overline{q_j q_{j+1}}$ at w after the nodal reduction.

Second, locally at q_0, our statements about Y_0 show that all branches of Υ_0 at q_0 are separated from each other after the nodal reduction. By that we mean among the components dominating Γ_1, Γ_2, $\overline{q_{-1} q_0}$ or $\overline{q_0 q_1}$ on $\widetilde{\Upsilon}_0$, no two are joined by a chain of curves contracting to q_0.

Third, locally at q_n $(n \neq 0)$, suppose that Γ_i passes through q_n and Y is the locally irreducible component of U containing the disk $\Delta(q_n)$. Then our statements about Y_0 show that $\overline{q_n q_{n+1}}$ (if $n < 0$) or $\overline{q_{n-1} q_n}$ (if $n > 0$) must lie on R_{3-i}. This also implies that Γ_1 and Γ_2 cannot both pass through q_n for any $n \neq 0$. Let $\widetilde{Y} = Y \times_\Upsilon \widetilde{\Upsilon}$. Then $\widetilde{Y}_0$ is

the union of the components of $\widetilde{U}_0$ which form a connected component of the dual graph of $\widetilde{U}_0$ corresponding to Y. By Proposition 2.4, $\widetilde{Y}_0$ consists of $\Delta(q_n)$ and curves over $\cup_{i=0}^{n+1}\overline{q_{i-1}q_i}$ (if $n < 0$) or $\cup_{i=1}^{n}\overline{q_{i-1}q_i}$ (if $n > 0$) and each $q_{i-1}q_i$ is dominated by the components of $\widetilde{Y}_0$ through maps whose total degree is $E \cdot \Delta(q_n)$. Later, we will prove in Theorem 4.1 that $E \cdot \Delta(q_n) = 1$, i.e., if Γ_i passes through q_n, Γ_i and E must meet transversely at q_n.

Proof of Proposition 2.4. Let us first prove the statements concerning Y_0. The rest will follow more or less immediately.

Obviously, there is at most one disk $\Delta \subset Y_0$ with $\Delta \subset \Gamma_1 \cup \Gamma_2$ by Proposition 2.3; otherwise, either Γ_i is joined to itself by a chain of curves over $\cup_{j=-l}^{m-1}\overline{q_jq_{j+1}}$ for some i or Γ_1 and Γ_2 are joined by a chain of curves over $\cup_{j=-l}^{m-1}\overline{q_jq_{j+1}}$, but by Proposition 2.3, Γ_1 and Γ_2 are already joined by the curves over a Type I chain.

If $\overline{q_iq_{i+1}} \subset Y_0$ for some $i \geq 0$, by Lemma 2.1, either $\overline{q_{i+1}q_{i+2}} \subset Y_0$ or there is a disk $\Delta(q_{i+1}) \subset \Gamma_1 \cup \Gamma_2$ such that $\Delta(q_{i+1}) \subset Y_0$ and $\Delta(q_{i+1})$ and $\overline{q_iq_{i+1}}$ lie on the different R_j's. If $\overline{q_{i+1}q_{i+2}} \not\subset Y_0$, we are done; otherwise, we apply the same argument to $\overline{q_{i+1}q_{i+2}}$ again. And eventually, this sequence of curves will end up at some disk $\Delta(q_n) \subset Y_0$ for some $n > i$. Obviously, $\overline{q_nq_{n+1}} \not\subset Y_0$; otherwise, we may continue to apply the above argument to show that there exists another disk $\Delta(q_{n'}) \subset Y_0$ for some $n' > n$. On the other hand, since $\overline{q_iq_{i+1}} \subset Y_0$, we have either $\overline{q_{i-1}q_i} \subset Y_0$ or $\Delta(q_{i-1}) \subset Y_0$ by Lemma 2.1. Since $\Delta(q_n) \subset Y_0$, we necessarily have $\overline{q_{i-1}q_i} \subset Y_0$. Apply the same argument to $\overline{q_{i-1}q_i}$ and we obtain that $\overline{q_{i-2}q_{i-1}} \subset Y_0$. So eventually, we have $\overline{q_0q_1}, \overline{q_1q_2}, ..., \overline{q_{n-1}q_n}, \Delta(q_n) \subset Y_0$. It is impossible that $\overline{q_{-1}q_0} \subset Y_0$; otherwise, we may apply the same line of argument to show subsequently that $\overline{q_{-1}q_0}, \overline{q_{-2}q_{-1}}, ..., \overline{q_{-l}q_{-l+1}} \subset Y_0$ and eventually $\Delta(q_{-l}) \subset Y_0$. So $\overline{q_{-1}q_0} \not\subset Y_0$, $\overline{q_nq_{n+1}} \not\subset Y_0$ and hence $(Y_0)_{\text{red}} = \cup_{j=0}^{n-1}\overline{q_jq_{j+1}} \cup \Delta(q_n)$. By Lemma 2.1, all $\overline{q_jq_{j+1}}$ in this sequence has the same multiplicity $\Delta(q_n) \cdot E$ in Y_0. So if $\overline{q_iq_{i+1}} \subset Y_0$ for some $i \geq 0$, we will necessarily end up in case (c).

The same argument shows if $\overline{q_iq_{i+1}} \subset Y_0$ for some $i < 0$, we will end up in case (d).

It is obvious that if there is no $\overline{q_iq_{i+1}} \subset Y_0$, we will necessarily end up in case (a) or (b).

So if $\overline{q_iq_{i+1}} \subset Y_0$ for $i \geq 0$, we necessarily have $\overline{q_jq_{j+1}} \subset Y_0$ for any $0 \leq j \leq i$ and $\overline{q_jq_{j+1}}$ has the same multiplicity in Y_0 as $\overline{q_iq_{i+1}}$. Therefore, $\mu(\overline{q_0q_1}) \geq \mu(\overline{q_1q_2}) \geq ... \geq \mu(\overline{q_{m-1}q_m})$. Similarly, $\mu(\overline{q_{-l}q_{-l+1}}) \leq \mu(\overline{q_{-l+1}q_{-l+2}}) \leq ... \leq \mu(\overline{q_{-1}q_0})$.

It follows from Corollary 2.1 that $m_1(q_0) \leq 1$ and $m_2(q_0) \leq 1$. Otherwise, suppose that $m_1(q_0) > 1$. Let Y be the locally irreducible component of Υ around $\cup_{i=-l}^{m-1}\overline{q_iq_{i+1}}$ such that $\Delta(q_0) \subset \Gamma_1$ and $\Delta(q_0) \subset Y_0$. By Corollary 2.1 and the fact that $\Delta(q_0)$ is the only "piece" of $\Gamma_1 \cup \Gamma_2$ in Y_0, we must have $\overline{q_{-1}q_0} \subset Y_0$, which contradicts our results on possible Y_0's.

Finally, it is impossible that $m_1(q_n) > 0$ and $m_2(q_n) > 0$ for $n \neq 0$. Otherwise, suppose that both Γ_1 and Γ_2 pass through q_n for some $n > 0$. Since Γ_1 passes through q_n, there exists a locally irreducible component Y of Υ around $\cup_{i=-l}^{m-1}\overline{q_iq_{i+1}}$ such that $(Y_0)_{\text{red}} = \cup_{i=0}^{n-1}\overline{q_iq_{i+1}}\cup\Delta(q_n)$ where $\Delta(q_n) \subset \Gamma_1$ and $\Delta(q_n)$ and $\overline{q_{n-1}q_n}$ lie on the different R_j's. So $\overline{q_{n-1}q_n}$ lies on R_2. The same argument shows that $\overline{q_{n-1}q_n}$ lies on R_1 since Γ_2 passes through q_n. Contradiction. $\square$

As Proposition 2.3 and 2.4 for Type I and Type II chains, we have a similar statement for Type III chains. However, we do not really need it in our proof. So we will state the proposition without a proof. Interested readers could follow the same line of argument as in the Proposition 2.3 and 2.4 and give a proof themselves.

Proposition 2.5. *A type III chain in Υ_0 containing the point r_{ij} must also contain the curve $\overline{r_{ij}}$.*

Let $\overline{r_{ij}} \cup \overline{r_{ij}q_1} \cup \overline{q_1q_2} \cup ... \cup \overline{q_{m-1}q_m}$ be a Type III chain in Υ_0. Assume that $r_{ij} = r_{11}$ without the loss of generality.

Let Y be a locally irreducible component of Υ around $\overline{r_{11}} \cup \overline{r_{11}q_1} \cup \overline{q_1q_2} \cup ... \cup \overline{q_{m-1}q_m}$. Let $q_0 = r_{11}$.

Then

1. *$2\mu(\overline{r_{11}}) \geq \mu(\overline{r_{11}q_1}) \geq \mu(\overline{q_1q_2}) \geq ... \geq \mu(\overline{q_{m-1}q_m})$;*
2. *$m_i(q_j)$ are even for all i and j;*
3. *$m_i(q_j) = 0$ if $i + j$ is odd;*
4. *Y_0 is one of the following*
   (a) *$Y_0 = \Delta(w)$ where w is one of the intersections between $\Gamma_1 \cup \Gamma_2$ and $\overline{r_{ij}} \cup \overline{r_{ij}q_1} \cup \overline{q_1q_2} \cup ... \cup \overline{q_{m-1}q_m}$ other than the points $r_{11}, q_1, q_2, ..., q_m$ and $\Delta(w)$ is an analytic neighborhood of w on Γ_1 or Γ_2;*
   (b) *$(Y_0)_{\text{red}} = \overline{r_{11}} \cup \overline{r_{11}q_1} \cup \overline{q_1q_2} \cup ... \cup \overline{q_{n-1}q_n} \cup \Delta(q_n)$ for some $n \geq 0$, where $\Delta(q_n)$ is an analytic neighborhood of q_n on Γ_1 or Γ_2, $\Delta(q_n)$ and $\overline{q_{n-1}q_n}$ lie on the different R_j's, $\overline{q_iq_{i+1}}$ has multiplicity $\Delta(q_n) \cdot E$ in Y_0 and $\overline{r_{11}}$ has multiplicity $(\Delta(q_n) \cdot E)/2$ in Y_0.*

3. Deformation around a Type I Chain

In this section, we will study the behavior of Υ_t at the neighborhood of a Type I chain in order to show that Υ_t has only nodes as singularities in the neighborhood of a Type I chain.

For a one-parameter family of curves S over Δ and a reduced subscheme $B \subset S_0$, we use the notation $\delta(S_t, B)$ to denote the total δ-invariant of the general fiber S_t in a neighborhood of B.

The main theorem of this section is

Theorem 3.1. *Let* $\cup_{i=1}^{2l} \overline{q_{i-1}q_i}$ *be the Type I chain in* Υ_0. *Assume that* $\overline{q_0 q_1} \subset R_1$. *Then*

1. $\delta(\Upsilon_t, \cup_{i=1}^{2l} \overline{q_{i-1}q_i}) = (2l+1)\beta - 1$, *where* $\beta = m_2(q_0) = m_1(q_{2l})$;
2. Υ_t *has exactly* β *nodes in the neighborhood of each point* w_j *for* $j = 1, 2, ..., 2l$, *where* w_j *is the intersection between* $\Gamma_1 \cup \Gamma_2$ *and* $\overline{q_{j-1}q_j}$ *other than the points* q_{j-1} *and* q_j;
3. Υ_t *has exactly* $\beta - 1$ *nodes in the neighborhood of* q_l.

Remark 3.1. Notice that

$$\delta(\Upsilon_t, \cup_{i=1}^{2l} \overline{q_{i-1}q_i}) = (2l+1)\beta - 1$$
$$= \sum_{i=0}^{2l} \text{Intsc}(\Sigma_1, E; q_i) - 1$$
$$= \sum_{i=0}^{2l} \text{Intsc}(\Sigma_2, E; q_i) - 1,$$

where $\text{Intsc}(A, B; C)$ denotes the local intersection multiplicities between A and B along C.

The main obstacle here is that Υ_0 is nonreduced along $\cup_{i=1}^{2l} \overline{q_{i-1}q_i}$ if $\beta > 1$. To study the deformation of a nonreduced curve, we introduce a method called "patching technique". Actually, this is a very commonly used method in deformation theory. To study the deformation of a projective variety or compact complex manifold M, we cover M by affine or analytic open sets, study the deformation of each piece separately and then "patch" these deformations together. We will show how this can be done for a nonreduced curve.

3.1. Deformation of a Nonreduced Planary Curve.

Definition 3.1. Let C be a nonreduced scheme and C_{red} be its reduced subscheme. Let $\mathcal{I}$ be the ideal sheaf of C_{red} in C. Then $\mathcal{N}_{C_{\text{red}}} = \mathcal{H}om(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_{C_{\text{red}}})$ is a coherent sheaf over C_{red}, which we will call the "intrinsic normal sheaf" of C.

Definition 3.2. We call a nonreduced curve C "planary" if it can be locally embedded to the 2-dimensional polydisk Δ_{xy}^2 everywhere. We say C has irreducible support if C_{red} is irreducible.

If C is a planary nonreduced curve with irreducible support, we can cover C with analytic open sets U_α such that each

$$U_\alpha \cong \operatorname{Spec} \mathbb{C}[[x, y]]/(f^m(x, y)),$$

where $f(x, y) = 0$ defines C_{red} in U_α and m is called the multiplicity of C. And it is obvious in this case that the intrinsic normal sheaf $\mathcal{N}_{C_{\mathrm{red}}}$ of C is a line bundle over C_{red}.

Obviously, C is planary if it lies on a smooth surface. If C lies on a smooth surface S and has irreducible support, its intrinsic normal sheaf is simply the normal sheaf $\mathcal{N}_{C_{\mathrm{red}}/S}$ of $C_{\mathrm{red}} \subset S$.

Now let C be a nonreduced planary curve with irreducible support and assume that C_{red} is smooth. Then C is locally isomorphic to $\operatorname{Spec} \mathbb{C}[[x, y]]/(y^m)$, where m is the multiplicity of C. Let $\mathcal{C}$ be a one-parameter family of curves over disk Δ whose central fiber is C. Cover $\mathcal{C}$ with open sets U_α of $\mathcal{C}$ such that $U_\alpha \cong \operatorname{Spec} \mathbb{C}[[x, y, t]]/(F_\alpha(x, y, t))$ where $F_\alpha(x, y, t)$ can be put into the form

$$F_\alpha(x, y, t) = y^m + \sum_{\substack{i>0 \\ 0 \le j \le m-2}} f_{\alpha ij}(x) t^i y^j.$$

For each α, we let

$$\gamma(\alpha) = \min\left\{\frac{i}{m - j} : f_{\alpha ij}(x) \not\equiv 0\right\}.$$

Alternatively, we may define $\gamma(\alpha)$ as the largest number such that

$$t^{m\gamma(\alpha)}\big| F_\alpha(x, t^{\gamma(\alpha)} y, t).$$

After a base change, we may assume that $\gamma(\alpha) \in \mathbb{Z}$ for each α. Then we may put $F_\alpha(x, y, t)$ into the form

$$F_\alpha(x, y, t) = y^m + \sum_{j=0}^{m-2} t^{(m-j)\gamma(\alpha)} \phi_{\alpha j}(x, t) y^j$$

where $\phi_0(x, 0), \phi_1(x, 0), ..., \phi_{m-2}(x, 0)$ are not all zeros by the choice of $\gamma(\alpha)$.

Next, we will "patch" U_α together. We need the following "patching" lemma.

Lemma 3.1 (Patching Lemma)**.** *Let $S_1, S_2 \subset \Delta_{xy}^2 \times \Delta_t$ be two families of curves over disk Δ_t. Suppose that S_i $(i = 1, 2)$ is cut out by*

$$y^m + \sum_{j=0}^{m-2} t^{(m-j)\gamma_i} \phi_{ij}(x, t) y^j = 0$$

in $\Delta_{xy}^2 \times \Delta_t$, where $\gamma_i \in \mathbb{Z}$, $\gamma_i > 0$, and $\phi_{i,0}(x, 0), \phi_{i,1}(x, 0), ..., \phi_{i,m-2}(x, 0)$ are not all zero. If there is an isomorphism between S_1 and S_2 which preserves the base and induces the identity map on the central fibers, then $\gamma_1 = \gamma_2$ and $\phi_{1j}(x, 0) = \phi_{2j}(x, 0)$ for $j = 0, 1, ..., m - 2$.

Lemma 3.1 is more or less obvious and we will leave its proof to the readers. It follows from Lemma 3.1 that all $\gamma(\alpha)$ are equal and for each j, $\{\phi_{\alpha j}(x, 0)\}$ defines a global section of the line bundle $\mathcal{N}_{C_{\mathrm{red}}}^{\otimes(m-j)}$, i.e., there exists $s \in H^0(\mathcal{N}_{C_{\mathrm{red}}}^{\otimes(m-j)})$ such that $s|_{U_\alpha} = \phi_{\alpha j}(x, 0)$, where $\mathcal{N}_{C_{\mathrm{red}}}$ is the intrinsic normal sheaf of C.

Let $\gamma = \gamma(\alpha)$. Then $\{y = t^\gamma = 0\}$ is a well-defined closed subscheme of $\mathcal{C}$ supported at the central fiber. Let $\widetilde{\mathcal{C}}$ be the blowup of $\mathcal{C}$ along the subscheme $\{y = t^\gamma = 0\}$. It is not hard to see that $\widetilde{\mathcal{C}}_0$ is a curve in the linear series $|\mathcal{O}_{\mathbb{P}}(m)|$ on $\mathbb{P} = \mathbb{P}(\mathcal{O}_{C_{\mathrm{red}}} \oplus \mathcal{N}_{C_{\mathrm{red}}})$. Now $\widetilde{\mathcal{C}}_0$ is "less" nonreduced than C, i.e., each component of $\widetilde{\mathcal{C}}_0$ has multiplicity strictly less than m in $\widetilde{\mathcal{C}}_0$ due to our choice of γ. This makes $\widetilde{\mathcal{C}}$ easier to investigate than $\mathcal{C}$.

3.2. **Outline of the Proof of Theorem 3.1.** Let I_j be the nonreduced component of Υ_0 supported on $\overline{q_{j-1}q_j}$ with multiplicity β. Obviously, I_j is a nonreduced planary curve with trivial intrinsic normal sheaf.

The proof of Theorem 3.1 is carried out by repeatedly blowing up Υ along $\cup I_j$. To be precise, we will construct a sequence of families

(3.1) $$\Upsilon^{(l)} \to \Upsilon^{(l-1)} \to ... \to \Upsilon^{(0)} = \Upsilon$$

where

1. the morphisms $\Upsilon^{(j)} \to \Upsilon^{(j-1)}$ are isomorphisms on the general fibers for $j = 1, 2, ..., l$;
2. on the central fiber, $\Upsilon_0^{(j)}$ contains $\widetilde{I}_1 \cup \widetilde{I}_2 \cup ... \cup \widetilde{I}_{j-1} \cup \widetilde{I}_j \cup I_{j+1} \cup ... \cup I_{2l-j} \cup \widetilde{I}_{2l-j+1} \cup \widetilde{I}_{2l-j+2} \cup ... \cup \widetilde{I}_{2l-1} \cup \widetilde{I}_{2l}$ for $j = 0, 1, 2, ..., l$, where $\widetilde{I}_1, \widetilde{I}_2, ..., \widetilde{I}_{2l}$ are the curves dominating $(I_1)_{\mathrm{red}} = \overline{q_0q_1}, (I_2)_{\mathrm{red}} = \overline{q_1q_2}, ..., (I_{2l})_{\mathrm{red}} = \overline{q_{2l-1}q_{2l}}$, respectively; recall from Proposition 2.3 that $\widetilde{I}_j$ dominates $\overline{q_{j-1}q_j}$ with a degree β map totally ramified at q_{j-1} and q_j;

3. let $w_{j1}, w_{j2}, ..., w_{j\beta}$ be the points over w_i in the map $\widetilde{I}_j \to (I_j)_{\mathrm{red}}$ for $j = 1, 2, ..., 2l$; there is a contractible curve H_j meeting $\widetilde{I}_j$ transversely at $w_{j1}, w_{j2}, ..., w_{j\beta}$ and meeting Γ_1 or Γ_2 transversely at another point;

4. each pair of curves $\widetilde{I}_j$ and $\widetilde{I}_{j+1}$ meet transversely at a point over q_j, which we still denote by q_j, except that $\widetilde{I}_l$ and $\widetilde{I}_{l+1}$ meet at the point q_l with multiplicity β on $\Upsilon^{(l)}$; more precisely, $\Upsilon^{(l)} \cong$ $\mathrm{Spec}\, \mathbb{C}[[y, z, t]]/(y(y + z^\beta + O(t)) - t^\alpha)$ in an analytic neighborhood of q_l, where we use the notation $O(f_1, f_2, ..., f_n)$ to denote an element generated by $f_1, f_2, ..., f_n$ in some ring (it should be clear from the context which ring we are talking about).

It is obvious from the above description that Υ has exactly β nodes in the neighborhoods of w_j. The statement that Υ_t has exactly $\beta - 1$ nodes in the neighborhood of q_l follows from a theorem of L. Caporaso and J. Harris on the deformation of tacnodes [CH, Lemma 4.1] and the fact that the curves $\widetilde{I}_l$ and $\widetilde{I}_{l+1}$ meet transversely on the nodal reduction $\widetilde{\Upsilon}$.

So the proof of Theorem 3.1 boils down to the construction of the blowup sequence (3.1).

3.3. **Construction of the Blowup Sequence.** We will do the construction inductively. For that purpose, we will work on an analytic neighborhood of Υ around $(\cup I_j)_{\mathrm{red}}$, which we will still refer to by Υ and can be described as follows.

First, Υ is a one-parameter family of curves over disk Δ_t with irreducible general fibers Υ_t. Second, $\Upsilon_0 = I_1 \cup I_2 \cup ... \cup I_{2l} \cup \Delta(q_0) \cup \Delta(q_{2l}) \cup \Delta(w_1) \cup \Delta(w_2) \cup ... \cup \Delta(w_{2l})$, where $I_j \cong \mathbb{P}^1 \times \mathrm{Spec}\, \mathbb{C}[z]/(z^\beta)$ and $\Delta(q_0)$, $\Delta(q_{2l})$ and $\Delta(w_i)$ are disks centered at q_0, q_{2l} and w_i regarded as the analytic neighborhoods of q_0, q_{2l} and w_i on Γ_1 or Γ_2. Finally, these curves are "patched" up in the following way:

1. $q_j = (I_j)_{\mathrm{red}} \cap (I_{j+1})_{\mathrm{red}}$ and

$$(3.2) \qquad \Upsilon \cong \mathrm{Spec}\, \mathbb{C}[[x, y, z, t]]/(xy - t^\alpha, z^\beta + O(t))$$

in the neighborhoods of q_j for $j = 1, ..., 2l - 1$,

2. $q_0 = \Delta(q_0) \cap (I_1)_{\mathrm{red}}$, $q_{2l} = \Delta(q_{2l}) \cap (I_{2l})_{\mathrm{red}}$ and

$$(3.3) \qquad \Upsilon \cong \mathrm{Spec}\, \mathbb{C}[[x, y, z, t]]/(xy - t^\alpha, y - z^\beta + O(t))$$

in the neighborhoods of q_0 and q_{2l};

3. $w_j = \Delta(w_j) \cap (I_j)_{\mathrm{red}}$ and

$$(3.4) \qquad \Upsilon \cong \mathrm{Spec}\, \mathbb{C}[[x, z, t]]/(xz^\beta + O(t))$$

in the neighborhoods of w_j for $j = 1, 2, ..., 2l$.

We also know that the nodal reduction $\widetilde{\Upsilon}$ of Υ has the properties described in Proposition 2.3 with $\Delta(q_0)$, $\Delta(q_{2l})$ and $\Delta(w_i)$ regarded as "pieces" of Γ_1 or Γ_2.

By (3.4), Υ can be locally embedded into $\Delta_{xz}^2 \times \Delta_t$ at w_j. After applying some automorphism of $\Delta_{xz}^2 \times \Delta_t$ which preserves the base and the central fiber, we may put its local defining equation at w_j into the form

$$(3.5) \qquad xz^\beta + t^{m_{1j}}z^{\beta-1} + \sum_{i=2}^{\beta} t^{m_{ij}} f_{ij}(x,t)z^{\beta-i} = 0$$

where $f_{ij}(x,t) \in \mathbb{C}[[x,t]]$, $f_{ij}(x,0) \neq 0$ and we put $m_{ij} = \infty$ if the corresponding term $z^{\beta-1}$ or $f_{ij}(x,t)z^{\beta-i}$ does not appear in the defining equation. Let

$$\gamma_j = \min\left\{\frac{m_{ij}}{i} : 1 \leq i \leq \beta\right\} \text{ and } \gamma = \min\left\{\frac{\alpha}{\beta}, \gamma_1, \gamma_2, ..., \gamma_{2l}\right\}.$$

Obviously, we may assume that $\gamma, \gamma_j \in \mathbb{Z}$ for $j = 1, 2, ..., 2l$ after a base change.

We can write (3.5) as

$$(3.6) \qquad xz^\beta + \sum_{i=1}^{\beta} t^{i\gamma} F_{ij}(x,t)z^{\beta-i} = 0$$

where $F_{ij}(x,t) \in \mathbb{C}[[x,t]]$ and $F_{1j}(x,t) = F_j(t) \in \mathbb{C}[[t]]$. Notice that $F_{ij}(x,0) = 0$ if $\gamma_j > \gamma$.

Claim 3.1. We claim that

1. $F_{ij}(0,0) = 0$ for $i = 1, 2, ..., \beta$ and $j = 1, 2, ..., 2l$ and especially, $F_{1j}(x,0) = F_j(0) = 0$;
2. $x^{-1}F_{ij}(x,0)$ extends to a meromorphic function G_{ij} on $(I_j)_{\text{red}}$;
3. each G_{ij} is holomorphic everywhere on $(I_j)_{\text{red}}$ except that $G_{\beta,1}$ and $G_{\beta,2l}$ have a simple pole at q_0 and q_{2l}, respectively, if $\alpha = \beta\gamma$;
4. at each q_j for $j = 1, 2, ..., 2l - 1$, $G_{i,j}(q_j) = G_{i,j+1}(q_j)$.

The first statement follows from the assumption that $\Delta(w_j)$ and $\widetilde{I}_j$ are disjoint on $\widetilde{\Upsilon}_0$.

Let U be an analytic neighborhood of w_j on Υ. Since $\Delta(w_j)$ and $\widetilde{I}_j$ are disjoint on $\widetilde{\Upsilon}_0$, U is reducible and $U = U^{(1)} \cup U^{(2)}$ after a base change where the central fibers of $U^{(1)}$ and $U^{(2)}$ are given by $x = t = 0$ and $z^\beta = t = 0$, respectively. Correspondingly, we may factor the LHS

of (3.6) in the following way

$$(3.7) \qquad xz^\beta + \sum_{i=1}^\beta t^{i\gamma} F_{ij}(x,t) z^{\beta-i} = (x + O(t))(z^\beta + O(t)).$$

It follows immediately from (3.7) that $F_{ij}(0,0) = 0$ which also implies $F_{1j}(x,0) = F_j(0) = 0$.

It follows from Lemma 3.1 that $x^{-1} F_{ij}(x,0)$ can be analytically extended to a section in

$$\Gamma\left((I_j)_{\mathrm{red}} \backslash \{q_{j-1}, q_j\}, \mathcal{N}_j^{\otimes i}\right)$$

where $\mathcal{N}_j$ is the intrinsic normal sheaf of I_j. Of course, $\mathcal{N}_j$ is trivial. So each $G_{ij} = x^{-1} F_{ij}(x,0)$ is a meromorphic function on $(I_j)_{\mathrm{red}} \cong \mathbb{P}^1$, which is holomorphic everywhere except at the points $q_0, q_1, q_2, ..., q_{2l}$.

Actually, G_{ij} can be extended over $q_1, q_2, ..., q_{2l-1}$. By (3.2), Υ can be locally embedded into $\mathrm{Spec}\,\mathbb{C}[[x,y,z,t]]/(xy - t^\alpha)$ at $q_1, q_2, ..., q_{2l-1}$ and its local defining equation at q_j can be put into the form

$$(3.8) \qquad z^\beta + \sum_{i=2}^\beta t^{i\delta_j} \phi_{ij}(x,y,t) z^{\beta-i} = 0$$

where $\delta_j > 0$, $\phi_{ij}(x,y,t) \in \mathbb{C}[[x,y,t]]$, for $i = 2, 3, ..., \beta$ and $j = 1, 2, ..., 2l-1$ and $\phi_{2j}(x,y,0), \phi_{3j}(x,y,0), ..., \phi_{\beta j}(x,y,0)$ do not all lie in the ideal $(xy) \subset \mathbb{C}[[x,y]]$.

By comparing (3.8) with (3.6) and applying Lemma 3.1, we have $\delta_j = \min(\gamma_j, \gamma_{j+1})$. So

1. $\delta_j \geq \gamma$;
2. $\gamma_j > \gamma$ and $\gamma_{j+1} > \gamma$ if $\delta_j > \gamma$.

Hence if $\delta_j > \gamma$, we have $G_{i,j} = G_{i,j+1} = 0$; otherwise, if $\delta_j = \gamma$, it is not hard to see that $G_{i,j}(q_j) = G_{i,j+1}(q_j) = \phi_{ij}(0,0,0)$. Hence G_{ij} are holomorphic at q_j and $G_{i,j}(q_j) = G_{i,j+1}(q_j)$ for $j = 1, 2, ..., 2l-1$.

So G_{ij}'s are holomorphic everywhere except at q_0 and q_{2l}. Next, we will try to find out what kind of singularities G_{ij}'s could have at q_0 and q_{2l}.

By (3.3), Υ can be locally embedded into $\mathrm{Spec}\,\mathbb{C}[[x,z,t]]$ at q_0 and q_{2l}. We can put its local defining equations at q_j into the form

$$(3.9) \qquad x\left(z^\beta + \sum_{i=2}^\beta t^{i\delta_j} \phi_{ij}(x,t) z^{\beta-i}\right) = t^\alpha$$

where $\delta_j > 0$, $\phi_{ij}(x,t) \in \mathbb{C}[[x,t]]$ for $i = 2, 3, ..., \beta$ and $j = 0, 2l$ and $\phi_{2j}(x,0), \phi_{3j}(x,0), ..., \phi_{\beta j}(x,0)$ are not all zero.

By comparing (3.9) with (3.6) and applying Lemma 3.1, we have

1. $\delta_j \geq \gamma$, $\alpha/\beta \geq \gamma_1$ and $\alpha/\beta \geq \gamma_{2l}$;
2. $G_{i,1}$ and $G_{i,2l}$ are holomorphic at q_0 and q_{2l}, respectively, for $i = 2, ..., \beta - 1$;
3. $G_{\beta,1}$ and $G_{\beta,2l}$ have a simple pole at q_0 and q_{2l}, respectively, if $\alpha = \beta\gamma$ and is holomorphic otherwise.

So we have justified every statement in Claim 3.1 and we are ready to construct a blowup map $\Upsilon' \to \Upsilon$ where $\Upsilon'_0 \supset \widetilde{I}_1 \cup I_2 \cup ... \cup I_{2l-1} \cup \widetilde{I}_{2l}$.

Notice that $z = t^\gamma = 0$ defines subschemes of Υ locally at w_j by (3.6). We can extend these subschemes and patch them together to obtain a closed subscheme of Υ due to our choice of the number γ. This subscheme is obviously supported on $(I_1)_{\mathrm{red}} \cup (I_2)_{\mathrm{red}} \cup ... \cup (I_{2l})_{\mathrm{red}}$. The family Υ' is obtained by blowing up Υ along this closed subscheme, which is locally cut out by $z = t^\gamma = 0$.

First we claim that $\alpha = \beta\gamma$. If not, we necessarily have $\alpha > \beta\gamma$. Hence each G_{ij} is a holomorphic function over I_j and consequently a constant. By the equality $G_{i,j-1}(q_j) = G_{i,j}(q_j)$, $G_{i1} = G_{i2} = ... = G_{i,2l}$ and we let $G_i = G_{i1}$.

Obviously, $\gamma_1 = \gamma_2 = ... = \gamma_{2l} = \gamma$. Otherwise, if $\gamma_k > \gamma$ for some k, then $G_{ik} = 0$, which implies $G_i = G_{ij} = 0$ for each i and j. And this is a contradiction to the choice of γ.

Let us examine the behavior of Υ' over the point q_0. Since $\gamma_1 = \gamma < \alpha/\beta$, by (3.9), Υ'_0 consists of the curve

$$(3.10) \qquad x\left(z_1^\beta + \sum_{i=2}^\beta G_i z_1^{\beta-i}\right) = 0,$$

where $z_1 = z/t^\gamma$. Obviously, the curve $x = 0$ dominates $\Delta(q_0)$ and the curve

$$(3.11) \qquad z_1^\beta + \sum_{i=2}^\beta G_i z_1^{\beta-i} = 0$$

maps to I_1. Since $G_2, G_3, ..., G_\beta$ are not all zero, the LHS of (3.11) has at least two distinct roots. Hence there are at least two different components over I_1 at q_0. This contradicts the fact that on $\widetilde{\Upsilon}_0$ there is a single component $\widetilde{I}_1$ dominating I_1 with a map totally ramified at q_0.

Therefore, we have $\alpha = \beta\gamma$. Then G_{ij} are still constants except that $G_{\beta,1}$ and $G_{\beta,2l}$ have a simple pole at q_0 and q_{2l}, respectively.

Let us examine the behavior of Υ' over the point q_0 again. Since $\alpha = \beta\gamma$, by our previous analysis, $\gamma = \gamma_1 = \alpha/\beta$. By (3.9), Υ'_0 consists

of the curve

$$(3.12) \qquad x\left(1 + \sum_{i=2}^{\beta-1} G_{i1}z_2^i\right) + (xG_{\beta,1})(q_0)z_2^\beta = zz_2 = 0$$

where $z_2 = t^\gamma/z$ and $(xG_{\beta,1})(q_0)$ reads as the value of the function $xG_{\beta,1}$ at point q_0. Obviously, the curve $x = z_2 = 0$ dominates $\Delta(q_0)$ and the irreducible curve

$$x\left(1 + \sum_{i=2}^{\beta-1} G_{i1}z_2^i\right) + (xG_{\beta,1})(q_0)z_2^\beta = z = 0$$

dominates $(I_1)_{\mathrm{red}}$ with a degree β map. So Υ_0' contains an irreducible curve $\widetilde{I}_1$ dominating $(I_1)_{\mathrm{red}}$ with a degree β map. And the map $\widetilde{I}_1 \to (I_1)_{\mathrm{red}}$ is totally ramified at q_0 and q_1.

Let us examine the behavior of Υ' over the point q_1, where Υ is defined by (3.8) in $\operatorname{Spec}\mathbb{C}[[x,y,z,t]]/(xy - t^\alpha)$. Without the loss of generality, we may assume that $I_1 \subset \{y = 0\}$ locally at q_1.

Since $G_{\beta,1}$ is a meromorphic function with a simple pole at q_1, $G_{\beta,1} \not\equiv 0$ in the neighborhood of q_1. Hence we must have $\phi_{\beta,1}(x,0,0) = G_{\beta,1}$ and $\delta_1 = \gamma$ in (3.8). Hence the curve Υ_0' is given by

$$(3.13) \qquad\qquad z_1^\beta + \sum_{i=2}^{\beta} G_{i1}z_1^{\beta-i} = xy = 0$$

in the neighborhood of q_1, where $z_1 = z/t^\gamma$. By (3.13), we necessarily have $G_{i1}(q_1) = 0$ for $i = 2, 3, ..., \beta$; otherwise, the map $\widetilde{I}_1 \to I_1$ will not be totally ramified at q_1. Therefore, $G_{i2} = G_{i2}(q_1) = G_{i1}(q_1) = 0$ and $\gamma_2 > \gamma$, which implies the curve in Υ_0' dominating I_2 is still a nonreduced curve isomorphic to $\mathbb{P}^1 \times \mathbb{C}[[z]]/(z^\beta)$. So we will use the same notation I_2 to denote the curve in Υ_0' over $I_2 \subset \Upsilon_0$.

The same argument can be carried out by studying the behavior of Υ' over $q_2, ..., q_{2l-1}$. Finally we obtained that $G_{ij} = 0$, $\gamma_j > \gamma$ and Υ_0' contains $I_j \cong \mathbb{P}^1 \times \mathbb{C}[[z]]/(z^\beta)$ for $i = 2, 3, ..., \beta$ and $j = 2, 3, ..., 2l - 1$. And by symmetry, Υ_0' contains the irreducible curve $\widetilde{I}_{2l}$ dominating I_{2l} with a degree β map. So we may take $\Upsilon^{(1)} = \Upsilon'$ in (3.1).

Next, we may take a neighborhood of Υ' around $I_2 \cup I_3 \cup ... \cup I_{2l-1}$ and go through this procedure again. Of course, we have to check that $\widetilde{I}_1, I_2, ..., I_{2l-1}, \widetilde{I}_{2l}$ are "patched" up at $q_1, q_2, ..., q_{2l-1}$ as required at the beginning of the construction.

Since $\widetilde{I}_1$ and $\widetilde{I}_{2l}$ are smooth everywhere, Υ' is locally given by (3.3) at q_1 and q_{2l-1}. It is easy to check that Υ' is locally given by (3.2) at $q_2, q_3, ..., q_{2l-2}$ and by (3.4) at $w_2, w_3, ..., w_{2l-1}$. So we may go through

the same procedure for $\Upsilon^{(1)} = \Upsilon'$ by blowing up Υ' along a subscheme of Υ_0' supported on $(I_2)_{\mathrm{red}} \cup (I_3)_{\mathrm{red}} \cup ... \cup (I_{2l-1})_{\mathrm{red}}$. The resulting family will be $\Upsilon^{(2)}$ and the blowup sequence (3.1) is constructed inductively in this way.

This finishes the construction of the blowup sequence (3.1) with only one thing left to check. We need to check that $\widetilde{I}_1$ and $\widetilde{I}_2$ meet at q_1 with multiplicity β if $l = 1$, i.e., we want to show that $\widetilde{I}_l$ and $\widetilde{I}_{l+1}$ meet at the point q_l with multiplicity β on $\Upsilon^{(l)}$.

Suppose that $l = 1$. By our previous argument, we still have $G_{i1} = G_{i2} = 0$ for $i = 2, 3, ..., \beta - 1$ and $G_{\beta,1}(q_1) = G_{\beta,2}(q_1) = 0$. Since $G_{\beta,1}$ and $G_{\beta,2}$ are meromorphic functions over $\mathbb{P}^1$ with exactly one simple pole at q_0 and q_2, respectively, each of them has a simple zero at q_1. Notice that $\phi_{\beta,1}(x, 0, 0)$ and $\phi_{\beta,1}(0, y, 0)$ are the localizations of $G_{\beta,1}$ and $G_{\beta,2}$ at q_1 in (3.8) since $\delta_1 = \gamma$. Hence $\phi_{\beta,1}(x, y, 0) = ax + by + O(x^2, xy, y^2)$ for some constants $a, b \neq 0$. Therefore, Υ' is cut out by

$$z_1^\beta + ax + by + O(x^2, xy, y^2, t) = 0$$

in $\mathrm{Spec}\,\mathbb{C}[[x, y, z_1, t]]/(xy - t^\alpha)$ locally at q_1.

4. Deformation Around a Type II Chain

Our main theorem of this section is the following.

Theorem 4.1. *Let $\cup_{i=-l}^{m-1} \overline{q_i q_{i+1}}$ be a Type II chain in Υ_0 and let $q_0 \in \{p_1, p_2, ..., p_{16}\}$. Then*

$$\delta\left(\Upsilon_t, \cup_{i=-l}^{m-1} \overline{q_i q_{i+1}}\right) \geq \sum_{i=-l}^{m} \mathrm{Intsc}(\Sigma_1, E; q_i).$$

If the equality holds, then
1. *$|\mu(\overline{q_{i-1} q_i}) - \mu(\overline{q_i q_{i+1}})| \leq 1$ for any i (let $\mu(q_i q_{i+1}) = 0$ if $i < -l$ or $i \geq m$);*
2. *all singularities of Υ_t are nodes in the neighborhood of $\cup_{i=-l}^{m-1} \overline{q_i q_{i+1}}$.*

4.1. **Some Basic Results on Curve Singularities.** Most of the following results on curve singularities are well known. But we will prove them here for the lack of a definite reference.

Proposition 4.1. *Let $C = \cup_{i=1}^{n} C_i$ be a reduced curve in Δ^2, where C_i are distinct curves in Δ^2. Then*

$$(4.1) \qquad \delta(C) = \sum_{i=1}^{n} \delta(C_i) + \sum_{1 \leq r < s \leq n} C_r \cdot C_s,$$

where $\delta(C)$ and $\delta(C_i)$ are the δ-invariants of C and C_i at the origin.

Proof. Let $\widetilde{\Delta^2}$ be the blowup of Δ^2 at the origin and let $\widetilde{C}$ and $\widetilde{C}_i$ be the proper transforms of C and C_i for $i = 1, 2, ..., n$. Let $\widetilde{C}$ meet the exception divisor at points $p_1, p_2, ..., p_l$ and let $\Delta^2(p_j)$ be the neighborhood of p_j in $\widetilde{\Delta^2}$.

Let $\widetilde{C}_i = \cup_{j=1}^l C_i^j$ where $C_i^j \subset \Delta^2(p_j)$ (let $C_i^j = \emptyset$ if $\widetilde{C}_i$ does not pass through p_j). Let m_i be the multiplicity of C_i at the origin and let $m = \sum_{i=1}^n m_i$ be the multiplicity of C at the origin. We argue by induction on $\delta(C)$. It is obvious when $\delta(C) = 0$. Suppose that $\delta(C) > 0$. Then $m > 1$.

First, we have

$$(4.2) \qquad \delta(C_i) = \frac{m_i(m_i - 1)}{2} + \delta(\widetilde{C}_i) = \frac{m_i(m_i - 1)}{2} + \sum_{j=1}^l \delta(C_i^j).$$

Second, by induction hypothesis, we have

$$(4.3) \qquad \begin{aligned} \delta(C) &= \frac{m(m-1)}{2} + \sum_{j=1}^l \delta\left(\cup_{i=1}^n C_i^j\right) \\ &= \frac{m(m-1)}{2} + \sum_{j=1}^l \left(\sum_{i=1}^n \delta(C_i^j) + \sum_{1 \le r < s \le n} C_r^j \cdot C_s^j\right). \end{aligned}$$

Finally, we have

$$(4.4) \qquad C_r \cdot C_s = m_r m_s + \sum_{j=1}^l C_r^j \cdot C_s^j.$$

Combining (4.2), (4.3) and (4.4), we obtain (4.1). □

The next is a parameterized version of Proposition 4.1.

Proposition 4.2. *Let* $\Upsilon = \cup_{i=1}^n \Upsilon^{(i)} \subset \Delta^2 \times \Delta_t$ *be a reduced flat family of curves over* Δ_t *where* $\Upsilon^{(i)}$ *are distinct flat families of curves over* Δ_t. *Then*

$$(4.5) \qquad \delta(\Upsilon_t) = \sum_{i=1}^n \delta(\Upsilon_t^{(i)}) + \sum_{1 \le r < s \le n} \Upsilon_t^{(r)} \cdot \Upsilon_t^{(s)},$$

where $\delta(\Upsilon_t)$ *and* $\delta(\Upsilon_t^{(i)})$ *are the total* δ-invariants of the general fibers of Υ and $\Upsilon^{(i)}$ and the intersection between $\Upsilon_t^{(r)}$ and $\Upsilon_t^{(s)}$ is taken on the general fiber of $\Delta^2 \times \Delta_t \to \Delta_t$.

Proof. After a base change, we may assume that each singular point of Υ_t is given by a section $p : \Delta_t \to \Delta^2 \times \Delta_t$. At each point $p = p(t)$, we

have

$$(4.6) \qquad \delta(\Upsilon_t, p) = \sum_{i=1}^{n} \delta(\Upsilon_t^{(i)}, p) + \sum_{1 \leq r < s \leq n} \left(\Upsilon_t^{(r)} \cdot \Upsilon_t^{(s)} \right)_p$$

by Proposition 4.1, where $\delta(\Upsilon_t, p)$ and $\delta(\Upsilon_t^{(i)}, p)$ are the δ-invariants of Υ_t and $\Upsilon_t^{(i)}$ at p and $\left(\Upsilon_t^{(r)} \cdot \Upsilon_t^{(s)} \right)_p$ is the local intersection number between $\Upsilon_t^{(r)}$ and $\Upsilon_t^{(s)}$ at p (take $\left(\Upsilon_t^{(r)} \cdot \Upsilon_t^{(s)} \right)_p = 0$ if $\Upsilon_t^{(r)}$ and $\Upsilon_t^{(s)}$ do not meet at p). Obviously, each intersection between $\Upsilon_t^{(r)}$ and $\Upsilon_t^{(s)}$ is necessarily a singularity of Υ_t. So summing (4.6) over all sections p of singularities yields (4.5). $\qquad \square$

Corollary 4.1. *Let* $\Upsilon = \cup_{i=1}^{n} \Upsilon^{(i)} \subset \Delta^2 \times \Delta_t$ *be a reduced flat family of curves over* Δ_t *where* $\Upsilon^{(i)}$ *are distinct flat families of curves over* Δ_t. *If* $\Upsilon_0^{(r)}$ *and* $\Upsilon_0^{(s)}$ *meet properly on the central fiber of* $\Delta^2 \times \Delta_t \to \Delta_t$ *for any* $1 \leq r < s \leq n$, *then*

$$(4.7) \qquad \delta(\Upsilon_t) \geq \sum_{1 \leq r < s \leq n} \Upsilon_0^{(r)} \cdot \Upsilon_0^{(s)},$$

where the intersection between $\Upsilon_0^{(r)}$ *and* $\Upsilon_0^{(s)}$ *is taken on the central fiber of* $\Delta^2 \times \Delta_t \to \Delta_t$.

Proof. Since $\Upsilon_0^{(r)}$ and $\Upsilon_0^{(s)}$ meet properly, $\Upsilon_0^{(r)} \cdot \Upsilon_0^{(s)} = \Upsilon_t^{(r)} \cdot \Upsilon_t^{(s)}$. Then (4.7) follows from (4.5). $\qquad \square$

The following is a special case of Corollary 4.1, which is directly applicable to our situation.

Corollary 4.2. *Let* $\Upsilon \subset \Delta^2 \times \Delta_t$ *be a reduced flat family of curves over* Δ_t *whose central fiber* Υ_0 *consists of* n *irreducible components* $\Gamma_1, \Gamma_2, ..., \Gamma_n$ *with multiplicities* $\mu_1, \mu_2, ..., \mu_n$, *respectively.*

Let $\pi : \widetilde{\Upsilon} \to \Upsilon$ *be the nodal reduction of* Υ *and let* $\widetilde{\Upsilon} = \cup_{i=1}^{\alpha} \widetilde{\Upsilon}^{(i)}$ *where* $\widetilde{\Upsilon}^{(i)}$ *are the connected components of* $\widetilde{\Upsilon}$. *Suppose that each* $\pi(\widetilde{\Upsilon}_0^{(i)})$ *is supported on* Γ_j *for some* $1 \leq j \leq n$. *Then*

$$(4.8) \qquad \delta(\Upsilon_t) \geq \sum_{1 \leq r < s \leq n} \mu_r \mu_s \left(\Gamma_r \cdot \Gamma_s \right).$$

Proof. By our assumption on $\widetilde{\Upsilon}$, we can write Υ as $\Upsilon = \cup_{j=1}^{n} \Upsilon^{(j)}$ where $\Upsilon_0^{(j)}$ consists of the component Γ_j with multiplicity μ_j, i.e., we let

$$\Upsilon^{(j)} = \pi \left(\bigcup_{\pi(\widetilde{\Upsilon}_0^{(i)})_{\mathrm{red}} \subset \Gamma_j} \widetilde{\Upsilon}^{(i)} \right).$$

Since $\Upsilon_0^{(r)}$ and $\Upsilon_0^{(s)}$ meet properly for any $r \neq s$, (4.8) follows from Corollary 4.1.   □

Proposition 4.3. *Let* $\Upsilon, \Upsilon_0, \Gamma_j, \mu_j, \pi, \widetilde{\Upsilon}$ *and* $\widetilde{\Upsilon}^{(i)}$ *be defined as in Corollary 4.2 except that we further assume each* $\pi(\widetilde{\Upsilon}_0^{(i)})$ *to be reduced, i.e.,* $\pi(\widetilde{\Upsilon}_0^{(i)}) = \Gamma_j$ *for some* j. *If*

$$(4.9) \qquad\qquad \delta(\Upsilon_t) = \sum_{1 \leq r < s \leq n} \mu_r \mu_s,$$

then Υ_t *only has nodes as singularities.*

Proof. By our assumptions on $\widetilde{\Upsilon}$, we have

$$\Upsilon = \bigcup_{i=1}^{n} \bigcup_{j=1}^{\mu_i} \Upsilon^{(i,j)}$$

where $\Upsilon_0^{(i,j)} = \Gamma_i$. By Proposition 4.2,

$$\delta(\Upsilon_t) = \sum_{i,j} \delta(\Upsilon_t^{(i,j)}) + \sum_{(r,p)<(s,q)} \Upsilon_t^{(r,p)} \cdot \Upsilon_t^{(s,q)}$$

where we define $(r, p) < (s, q)$ if either $r < s$ or $r = s$ and $p < q$. Notice that

$$\sum_{(r,p)<(s,q)} \Upsilon_t^{(r,p)} \cdot \Upsilon_t^{(s,q)} \geq \sum_{p,q} \sum_{r<s} \Upsilon_t^{(r,p)} \cdot \Upsilon_t^{(s,q)}$$

$$= \sum_{p,q} \sum_{r<s} \Upsilon_0^{(r,p)} \cdot \Upsilon_0^{(s,q)} = \sum_{r<s} \mu_r \mu_s \left(\Gamma_r \cdot \Gamma_s \right)$$

$$\geq \sum_{r<s} \mu_r \mu_s.$$

If the equality holds, we will necessarily have that $\delta(\Upsilon_t^{(i,j)}) = 0$, $\Upsilon_t^{(r,p)} \cdot \Upsilon_t^{(s,q)} = 0$ if $r = s$ and 1 if $r \neq s$. This implies that $\Upsilon_t^{(r,p)}$ and $\Upsilon_t^{(s,q)}$ meet transversely for any $r \neq s$ and $\Upsilon_t^{(r,p)} \cap \Upsilon_t^{(s,q)} \neq \Upsilon_t^{(r',p')} \cap \Upsilon_t^{(s',q')}$ for any $r < s$, $r' < s'$ and $(p, q, r, s) \neq (p', q', r', s')$. Therefore, Υ_t has exactly $\sum_{r<s} \mu_r \mu_s$ nodes as singularities, which are the intersections $\Upsilon_t^{(r,p)} \cap \Upsilon_t^{(s,q)}$ for $r < s$.   □

Proposition 4.4. *Let $X \subset \Delta^3_{xyz} \times \Delta_t$ be defined by $xy = t^\alpha z$ for some $\alpha > 0$ and let $X_0 = R_1 \cup R_2$ and $E = R_1 \cap R_2$, where $R_1 = \{x = t = 0\}$ and $R_2 = \{y = t = 0\}$.*

Let $\Upsilon \subset X$ be a reduced flat family of curves whose central fiber Υ_0 consists of $m+n$ irreducible components $\Gamma^{(1)}_1, \Gamma^{(1)}_2, ..., \Gamma^{(1)}_m, \Gamma^{(2)}_1, \Gamma^{(2)}_2, ...\Gamma^{(2)}_n$ where $\Gamma^{(i)}_j \subset R_i$ and $\Gamma^{(i)}_j \neq E$. Let μ_{ij} be the multiplicity of $\Gamma^{(i)}_j$ in Υ_0.

Let $\pi : \widetilde{\Upsilon} \to \Upsilon$ be the nodal reduction of Υ and let $\widetilde{\Upsilon} = \cup \widetilde{\Upsilon}^{(k)}$ where $\widetilde{\Upsilon}^{(k)}$ are the connected components of $\widetilde{\Upsilon}$. Suppose that each $\pi(\widetilde{\Upsilon}^{(k)}_0)$ is supported on $\Gamma^{(i)}_j$ for some i and j. Then

$$(4.10) \qquad \delta(\Upsilon_t) \geq \left(\sum_j \mu_{ij} \left(\Gamma^{(i)}_j \cdot E \right) \right) \left(\sum_j \mu_{3-i,j} \right),$$

for $i = 1, 2$.

Proof. We resolve the double point p in the same way as in the proof of Corollary 2.1. Let $\widetilde{X} \subset X \times \mathbb{P}^1$ be defined by

$$\frac{x}{t^\alpha} = \frac{z}{y} = \frac{W_1}{W_0}$$

where (W_0, W_1) are the homogeneous coordinates of $\mathbb{P}^1$. Let $\widetilde{X}_0 = \widetilde{R_1} \cup R_2$ and $\widetilde{E} = \widetilde{R_1} \cap R_2$, where $\widetilde{R_1}$ is the blowup of R_1 at the origin p and $\widetilde{E} = \{x = y = t = W_0/W_1 = 0\}$. Let $\widetilde{p} = \{x = y = z = t = W_0/W_1 = 0\}$ and let $P = \{x = y = z = t = 0\}$ be the exceptional curve of $\widetilde{X} \to X$. Let $\widetilde{\Gamma^{(1)}_j}$ be the proper transform of $\Gamma^{(1)}_j$ and $p_j = \widetilde{\Gamma^{(1)}_j} \cap P$ for $j = 1, 2, ..., m$. Our assumptions on $\widetilde{\Upsilon}$ guarantee $p_j \neq \widetilde{p}$; otherwise, $\widetilde{\Gamma^{(1)}_j}$ will pass through $\widetilde{p}$ and by Lemma 2.1, each component of $\widetilde{\Upsilon}_0$ that dominates $\widetilde{\Gamma^{(1)}_j}$ will be joined by a chain of curves to a component dominating $\Gamma^{(2)}_k$ for some k.

Let $\mathcal{Y} = \Upsilon \times_X \widetilde{X}$, $\widetilde{\mathcal{Y}} = \widetilde{\Upsilon} \times_X \widetilde{X}$ and $\widetilde{\pi} : \widetilde{\mathcal{Y}} \to \mathcal{Y}$ be the map induced by π. By Lemma 2.1, $\mathcal{Y}_0$ contains P with multiplicity

$$\sum_{j=1}^n \mu_{2j} \left(\Gamma^{(2)}_j \cdot E \right).$$

Let $\widetilde{\mathcal{Y}} = \cup \widetilde{\mathcal{Y}}^{(k)}$ where $\widetilde{\mathcal{Y}}^{(k)} = \widetilde{\Upsilon}^{(k)} \times_X \widetilde{X}$.

It is not hard to see that $\widetilde{\pi}(\widetilde{\mathcal{Y}}^{(k)}_0)_{\mathrm{red}} \subset \widetilde{\Gamma^{(1)}_j} \cup P$ if $\pi(\widetilde{\Upsilon}^{(k)}_0)_{\mathrm{red}} \subset \Gamma^{(1)}_j$ for some j. By Lemma 2.1, $P \not\subset \widetilde{\pi}(\widetilde{\mathcal{Y}}^{(k)}_0)$ if $\pi(\widetilde{\Upsilon}^{(k)}_0)_{\mathrm{red}} \subset \Gamma^{(1)}_j$ for some j. Therefore, we may apply Corollary 4.2 to each points p_j for $j =$

$1, 2, ..., m$ and obtain

$$\delta(\Upsilon_t) = \delta(\mathcal{Y}_t) \geq \sum_{j=1}^{m} \delta(\mathcal{Y}_t, p_j)$$

$$\geq \sum_{j=1}^{m} \left(\mu_{1j} \sum_{l=1}^{n} \mu_{2l} \left(\Gamma_l^{(2)} \cdot E \right) \right)$$

$$= \left(\sum_{l=1}^{n} \mu_{2l} \left(\Gamma_l^{(2)} \cdot E \right) \right) \left(\sum_{j=1}^{m} \mu_{1j} \right).$$

Our argument needs some trivial change if p_j's fail to be distinct, which we will leave it to the readers. □

Proposition 4.5. *Let* $X, \Upsilon, \Gamma_j^{(i)}, \mu_{ij}, \pi, \widetilde{\Upsilon}$ *and* $\widetilde{\Upsilon}^{(k)}$ *be defined as in Proposition 4.4 except that we further assume each* $\pi(\widetilde{\Upsilon}_0^{(k)})$ *to be reduced, i.e.,* $\pi(\widetilde{\Upsilon}_0^{(k)}) = \Gamma_j^{(i)}$ *for some* i *and* j. *If*

$$(4.11) \qquad \delta(\Upsilon_t) = \left(\sum_{j=1}^{m} \mu_{1j} \right) \left(\sum_{j=1}^{n} \mu_{2j} \right),$$

then Υ_t *only has nodes as singularities.*

Proof. The proof is the same as the proof of Proposition 4.4 except that we need to apply Proposition 4.3 to each point p_j at the last step. □

4.2. **Proof of Theorem 4.1.** Without the loss of generality, let us assume that $\overline{q_0 q_1} \subset R_1$. Let $\alpha_1 = m_1(q_0)$, $\alpha_2 = m_2(q_0)$ and $\mu_i = \mu(\overline{q_i q_{i+1}})$ for each i.

Let w_i be the intersection between $\overline{q_i q_{i+1}}$ and $\Gamma_1 \cup \Gamma_2$ other than the points q_i and q_{i+1}, if such intersection exists.

Basically, Υ_t has singularities in the neighborhoods of q_0 and w_i. Proposition 2.4 tells us the configurations of $\widetilde{\Upsilon}$ over these neighborhoods, while the series of results we have obtained in 4.1 can be used to estimate the δ-invariants of Υ_t in these neighborhoods.

By Proposition 2.4 and 4.4,

$$(4.12) \qquad \delta(\Upsilon_t, q_0) \geq (\alpha_1 + \mu_0)^2 = (\alpha_2 + \mu_{-1})^2.$$

If $\alpha_1 = 0$, the point w_0 exists. By Proposition 2.4 and Corollary 4.2,

$$(4.13) \qquad \delta(\Upsilon_t, w_0) \geq \mu_0$$

if $\alpha_1 = 0$. Since $\alpha_1 = 0$ or 1, we may write (4.13) in the form

$$(4.14) \qquad \delta(\Upsilon_t, w_0) \geq (1 - \alpha_1)\mu_0.$$

Similarly, we have

(4.15) $$\delta(\Upsilon_t, w_{-1}) \geq (1 - \alpha_2)\mu_{-1}.$$

Obviously, for $i > 0$, the point w_i exists if $\mu_{i-1} = \mu_i$. By Proposition 2.4 and Corollary 4.2, for $i > 0$,

(4.16) $$\delta(\Upsilon_t, w_i) \geq \mu_i$$

if $\mu_{i-1} = \mu_i$. Similarly, for $i > 1$,

(4.17) $$\delta(\Upsilon_t, w_{-i}) \geq \mu_{-i}$$

if $\mu_{-i} = \mu_{-i+1}$.

Let $0 \leq a_0 < a_1 < a_2 < ... < a_n < ...$ be the sequence of integers such that

$$\mu_0 = ... = \mu_{a_0} > \mu_{a_0+1} = \mu_{a_0+2} = ... = \mu_{a_1} > \mu_{a_1+1} = \mu_{a_1+2} = ... = \mu_{a_2}$$
$$> ... > \mu_{a_{n-1}+1} = \mu_{a_{n-1}+2} = ... = \mu_{a_n} > ....$$

Then by (4.16),

(4.18) $$\sum_{i>0} \delta(\Upsilon_t, w_i) \geq a_0\mu_0 + \sum_{i>0}(a_i - a_{i-1} - 1)\mu_{a_i}.$$

And since

$$\sum_{i>0} \mathrm{Intsc}(\Sigma_1, E; q_i) = (a_0 + 1)\mu_0 + \sum_{i>0}(a_i - a_{i-1})\mu_{a_i},$$

(4.19)

$$\sum_{i>0} \delta(\Upsilon_t, w_i) - \sum_{i>0} \mathrm{Intsc}(\Sigma_1, E; q_i) \geq -\left(\mu_0 + \sum_{i>0}\mu_{a_i}\right)$$

$$\geq -(\mu_0 + (\mu_0 - 1) + (\mu_0 - 2) + ... + 1) = -\frac{\mu_0(\mu_0 + 1)}{2}.$$

Similarly,

(4.20) $$\sum_{i>1} \delta(\Upsilon_t, w_{-i}) - \sum_{i\geq 1} \mathrm{Intsc}(\Sigma_1, E; q_{-i}) \geq -\frac{\mu_{-1}(\mu_{-1} + 1)}{2}.$$

Combining (4.12), (4.14), (4.15), (4.19) and (4.20), we get

(4.21)

$$\delta(\Upsilon_t, \cup_{i=-l}^{m-1}\overline{q_iq_{i+1}}) - \sum_{i=-l}^{m} \mathrm{Intsc}(\Sigma_1, E; q_i)$$

$$\geq \delta(\Upsilon_t, q_0) + \left(\sum_{i=0,-1} \delta(\Upsilon_t, w_i) - \mathrm{Intsc}(\Sigma_1, E; q_0)\right)$$

$$+ \left(\sum_{i>0} \delta(\Upsilon_t, w_i) - \sum_{i>0} \mathrm{Intsc}(\Sigma_1, E; q_i) \right)$$

$$+ \left(\sum_{i>1} \delta(\Upsilon_t, w_{-i}) - \sum_{i\geq 1} \mathrm{Intsc}(\Sigma_1, E; q_{-i}) \right)$$

$$\geq \left(\frac{1}{2}(\alpha_1 + \mu_0)^2 + \frac{1}{2}(\alpha_2 + \mu_{-1})^2 \right)$$

$$+ \left((1 - \alpha_1)\mu_0 + (1 - \alpha_2)\mu_{-1} - \frac{1}{2}(\alpha_1 + \mu_0) - \frac{1}{2}(\alpha_2 + \mu_{-1}) \right)$$

$$- \frac{\mu_0(\mu_0 + 1)}{2} - \frac{\mu_{-1}(\mu_{-1} + 1)}{2}$$

$$= \frac{1}{2}(\alpha_1^2 - \alpha_1) + \frac{1}{2}(\alpha_2^2 - \alpha_2) = 0.$$

Suppose that the equality in (4.21) holds. Then the equalities in (4.19) and (4.20) have to hold, which implies $|\mu_{i-1} - \mu_i| = 0$ or 1 for any i. Under this condition, it is not hard to see by Proposition 2.4 that each component of $\widetilde{\Upsilon}_0$ dominating a component $\overline{q_i q_{i+1}} \subset \cup_{i=-l}^{m-1} \overline{q_i q_{i+1}}$ must dominate $\overline{q_i q_{i+1}}$ with a degree one map. So we may apply Proposition 4.3 to Υ at w_i and apply Proposition 4.5 to $\Upsilon \subset X$ at q_0 to conclude that Υ_t only has nodes as singularities in the neighborhoods of w_i and q_0. On the other hand, since the equality in (4.21) holds, Υ_t does not have singularities anywhere else in the neighborhood of $\cup_{i=-l}^{m-1} \overline{q_i q_{i+1}}$. Therefore, Υ_t only has nodes as singularities in the neighborhood of $\cup_{i=-l}^{m-1} \overline{q_i q_{i+1}}$ if the equality in (4.21) holds.

5. Deformation Around a Type III Chain

The deformation of Υ around a Type III chain is much more complicated than the other two cases. Instead of attempting a direct analysis as we did in the previous two sections, we will employ a completely different approach. Our strategy can be briefly described as follows.

We are trying to move $[R^s]$ around in $\widetilde{\mathcal{H}}_g$, or equivalently, apply a monodromy action to $[R^s]$ such that a limiting rational curve Υ_0 on R^s will become another limiting rational curve Υ_0' and meanwhile a Type III chain in Υ_0 will be "transformed" to a Type II chain in Υ_0'.

Our main theorem of this section is the following.

Theorem 5.1. *Let* $\overline{r_{ij}} \cup \overline{r_{ij}q_1} \cup ... \cup \overline{q_{m-1}q_m}$ *be a Type III chain in* Υ_0. *Then*

$$\delta(\Upsilon_t, \overline{r_{ij}} \cup \overline{r_{ij}q_1} \cup ... \cup \overline{q_{m-1}q_m}) \geq \sum_{l=0}^{m} \mathrm{Intsc}(\Sigma_1, E; q_l)$$

(let $q_0 = r_{ij}$*) and if the equality holds, all singularities of* Υ_t *are nodes in the neighborhood of* $\overline{r_{ij}} \cup \overline{r_{ij}q_1} \cup ... \cup \overline{q_{m-1}q_m}$.

5.1. **Notations and Definitions.**

In order to make our ideas precise, we need to introduce a few new objects.

Since we assume that $\widetilde{\mathcal{W}}_g$ is irreducible, C has the same number of singularities for all general $([S], [C]) \in \widetilde{\mathcal{W}}_g$. Let C have exactly β singular points for a general $([S], [C]) \in \widetilde{\mathcal{W}}_g$.

Let $\mathcal{Z}_g$ be the incidence correspondence

$$\mathcal{Z}_g = \Big\{ ([S], [C], w_1, w_2, ..., w_\beta, \delta_1, \delta_2, ..., \delta_\beta, \nu_1, \nu_2, ..., \nu_\beta) :$$
$$([S], [C]) \in \mathcal{W}_g \text{ general,}$$
$$C_{\mathrm{sing}} = \{w_1, w_2, ..., w_\beta\},$$
$$\delta_i \text{ is } \delta\text{-invariant of } C \text{ at } w_i,$$
$$\nu_i = 1 \text{ if } w_i \text{ is a node of } C; \nu_i = 0 \text{ if it is not} \Big\}$$
$$\subset \widetilde{\mathcal{W}}_g \times (\mathbb{P}^{4g-3})^\beta \times \mathbb{Z}^{2\beta}.$$

Again, we first define $\mathcal{Z}_g$ for general K3 surfaces S and then take its closure $\widetilde{\mathcal{Z}}_g$ in $\widetilde{\mathcal{W}}_g \times (\mathbb{P}^{4g-3})^\beta \times \mathbb{Z}^{2\beta}$.

Notice that the last 2β coordinates $(\delta_1, \delta_2, ..., \delta_\beta, \nu_1, \nu_2, ..., \nu_\beta)$ are the same throughout an irreducible component of $\widetilde{\mathcal{Z}}_g$. Theorem 1.1 is equivalent to the statement that $\nu_1 = \nu_2 = ... = \nu_\beta = 1$ on each irreducible component of $\widetilde{\mathcal{Z}}_g$ which dominates $\widetilde{\mathcal{H}}_g$. Without the loss of generality, let us assume that $\widetilde{\mathcal{Z}}_g$ is irreducible and dominates $\widetilde{\mathcal{H}}_g$.

Definition 5.1. *Let* $([S], [C], w_1, w_2, ..., w_\beta, ...) \in \widetilde{\mathcal{Z}}_g$. *We will call* w_i*'s "limiting singularities" of* C.

We observe that

Proposition 5.1. *Let* $[R^s] \in \widetilde{\mathcal{R}}_g$ *general and let* Υ_0 *be a limiting rational curve on* R^s. *Then each limiting singularity of* Υ_0 *lies on an F-chain in* Υ_0.

Proof. Notice that Υ_0 is smooth outside of its F-chains. $\qquad\square$

The same holds for a general $[R^s] \in \widetilde{\mathcal{R}}_g^0$.

Proposition 5.2. *Let* $[R^s] \in \widetilde{\mathcal{R}}_g^0$ *general and let* Υ_0 *be a limiting rational curve on* R^s. *Then each limiting singularity of* Υ_0 *is either*

a point on a component $\overline{pq} \subset \Upsilon_0, \overline{pq} \in |\mathcal{O}_R(F)|$ *or an point among* $\Gamma_1 \cap E = \Gamma_2 \cap E$.

Proof. Again, the reason for this fact is trivially that Υ_0 is smooth outside of the locus described. □

Let us consider the K3 surfaces S with Picard lattice

$$(5.1) \qquad\qquad \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$$

or

$$(5.2) \qquad\qquad \begin{pmatrix} -2 & 2 \\ 2 & 0 \end{pmatrix}$$

or equivalent, K3 surfaces whose Picard groups are generated by two divisors C and F with $C \cdot F = 2$, $F^2 = 0$ and $C^2 = 0$ or $C^2 = -2$. For lacking a good name for such surfaces, we will simply call them *elliptic K3*'s. Although elliptic K3 surfaces usually refer to all K3 surfaces that admit an elliptic fiberation, we will use the term in our context to refer to K3 surfaces with Picard lattice (5.1) or (5.2). It is not hard to see that such K3 surfaces S can be realized as double covers of $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{F}_1$ ramified along a curve in $|4C + 4F|$ or $|4C + 6F|$.

Notice that the divisor $C + kF$ is ample but not very ample on an elliptic K3 surface S. To embed S into a projective space, we need to use the divisor $2C + 2kF$. Embed S into $\mathbb{P}^{4g-3}$ $(g \geq 3)$ by $|2C + 2kF|$ and we observe that $[S] \in \mathcal{H}_g$, where $g = 2k + 1$ if S has Picard lattice (5.1) and $g = 2k$ if S has Picard lattice (5.2).

Let $\mathcal{Y}_g \subset \mathcal{H}_g$ be the locus in $\mathcal{H}_g$ consisting of elliptic K3's with Picard lattice (5.1) or (5.2) embedded into $\mathbb{P}^{4g-3}$ by $|2C + 2kF|$, where $k = \lfloor g/2 \rfloor$. And let $\widetilde{\mathcal{Y}_g}$ be the closure of $\mathcal{Y}_g$ in $\widetilde{\mathcal{H}_g}$, where we may regard $\mathcal{Y}_g$ as a subscheme of $\widetilde{\mathcal{H}_g}$ since $\mathcal{Y}_g$ is disjoint from $\mathcal{R}_g$. Obviously, $\mathcal{Y}_g$ and $\widetilde{\mathcal{Y}_g}$ are irreducible and have codimension 1 in $\mathcal{H}_g$ and $\widetilde{\mathcal{H}_g}$, respectively. And

Proposition 5.3. *We have* $\widetilde{\mathcal{Y}_g} \cap \widetilde{\mathcal{R}_g} = \widetilde{\mathcal{R}_g^0}$.

Proof. We realize $S \in \mathcal{Y}_g$ as double covers of $\mathbb{P} = \mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{F}_1$ totally ramified along a curve $C \in |-2K_{\mathbb{P}}|$, where $K_{\mathbb{P}}$ is the canonical divisor of $\mathbb{P}$. We want to construct a double cover Y of $X = |-2K_{\mathbb{P}}| \times \mathbb{P}$ whose restriction to a point $C \in |-2K_{\mathbb{P}}|$ is the double cover of $\mathbb{P}$ ramified along C.

Let $B \subset X = |-2K_{\mathbb{P}}| \times \mathbb{P}$ be the universal family of curves in $|-2K_{\mathbb{P}}|$ and let π_1 and π_2 be the projections of X to $|-2K_{\mathbb{P}}|$ and $\mathbb{P}$, respectively. Since

$$\mathcal{O}_X(B) = \pi_1^* \mathcal{O}_{|-2K_{\mathbb{P}}|}(1) \times \pi_2^* \mathcal{O}_{\mathbb{P}}(-2K_{\mathbb{P}}) \neq L^2$$

for any line bundle L, we cannot construct Y directly as a double cover of X along B. The remedy for this situation is trivial. We take a general N-dimensional linear subsystem of $|\mathcal{O}_{|-2K_\mathbb{P}|}(2)|$, where $N = \dim |-2K_\mathbb{P}|$, and use it to map $|-2K_\mathbb{P}|$ to itself. For example, after fixing homogeneous coordinates $(Z_0, Z_1, ..., Z_N)$ of $|-2K_\mathbb{P}|$, we may simply take the map to be sending $(Z_0, Z_1, ..., Z_N)$ to $(Z_0^2, Z_1^2, ..., Z_N^2)$. This will induce a map $f : X \to X$. Obviously,

$$f^*\mathcal{O}_X(B) = \pi_1^*\mathcal{O}_{|-2K_\mathbb{P}|}(2) \times \pi_2^*\mathcal{O}_\mathbb{P}(-2K_\mathbb{P})$$
$$= \left(\pi_1^*\mathcal{O}_{|-2K_\mathbb{P}|}(1) \times \pi_2^*\mathcal{O}_\mathbb{P}(-K_\mathbb{P})\right)^2.$$

So there exists a double cover Y of X ramified along $f^{-1}(B)$. It is trivial to check that the fiber of $Y \to |-2K_\mathbb{P}|$ over an irreducible $C \in |-2K_\mathbb{P}|$ is the double cover of $\mathbb{P}$ ramified along C, while the fiber of $Y \to |-2K_\mathbb{P}|$ over a double curve $C = 2D$ with $D \in |-K_\mathbb{P}|$ is a surface $R \in \mathcal{R}_g^0$. Therefore, $\mathcal{R}_g^0$ lies on the closure of $\mathcal{Y}_g$ in $\mathcal{H}_g$.

Notice that both $\widetilde{\mathcal{Y}_g}$ and $\widetilde{\mathcal{R}_g}$ has codimension 1 in $\widetilde{\mathcal{H}_g}$. Obviously, $\widetilde{\mathcal{Y}_g} \cap \widetilde{\mathcal{R}_g} \subset \widetilde{\mathcal{R}_g^0}$. The previous argument shows that $\widetilde{\mathcal{Y}_g} \cap \widetilde{\mathcal{R}_g} \neq \emptyset$. And since $\widetilde{\mathcal{R}_g^0}$ is irreducible with codimension 2 in $\widetilde{\mathcal{H}_g}$, $\widetilde{\mathcal{Y}_g} \cap \widetilde{\mathcal{R}_g} = \widetilde{\mathcal{R}_g^0}$. $\square$

5.2. **Rational Curves on Elliptic K3's.** As mentioned at the beginning of this section, our strategy is to move $[R^s]$ around in $\widetilde{\mathcal{H}_g}$. The way we move $[R^s]$ can now be described as follows:

(5.3) $[R^s] \in \widetilde{\mathcal{R}_g} \Rightarrow [R^s] \in \widetilde{\mathcal{R}_g^0} \Rightarrow [S] \in \mathcal{Y}_g \Rightarrow [R^s] \in \widetilde{\mathcal{R}_g^0} \Rightarrow [R^s] \in \widetilde{\mathcal{R}_g}.$

Namely, we first degenerate a R^s in $\widetilde{\mathcal{R}_g}$ to a R^s in $\widetilde{\mathcal{R}_g^0}$, then move a R^s in $\widetilde{\mathcal{R}_g^0}$ to an S in $\mathcal{Y}_g$ and so on.

It is clear what happens to the limiting rational curves on a R^s in $\widetilde{\mathcal{R}_g}$ when it degenerates to a R^s in $\widetilde{\mathcal{R}_g^0}$. However, we do not have much idea about what happens to the rational curves on an S in $\mathcal{Y}_g$ when it degenerates to a R^s in $\widetilde{\mathcal{R}_g^0}$. So we will prove a collection of results regarding rational curves on an elliptic K3 and their degeneration as an elliptic K3 degenerates to a special union of scrolls.

On each S in $\mathcal{Y}_g$, there is a pencil of elliptic curves in $|F|$. There are exactly 24 rational curves in $|F|$ for S general. We are interested in their monodromy group as S varies in $\mathcal{Y}_g$.

Proposition 5.4. *The monodromy group of the* 24 *rational curves in* $|F|$ *as S varies in $\mathcal{Y}_g$ is the full symmetric group.*

Proof. We may realize S as the double cover of $\mathbb{P} = \mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{F}_1$ ramified along a curve $C \in |-2K_\mathbb{P}|$, where $K_\mathbb{P}$ is the canonical divisor of $\mathbb{P}$. Let $\pi : S \to \mathbb{P}$ be the covering map. We use the same notation F

to denote the divisor on $\mathbb{P}$ such that $\pi^*(F) = F$ on S. Obviously, the 24 rational curves in $|\mathcal{O}_S(F)|$ correspond to the 24 curves in $|\mathcal{O}_\mathbb{P}(F)|$ tangent to C. Then the statement of the proposition is equivalent to saying that the monodromy group G of the 24 curves in $|\mathcal{O}_\mathbb{P}(F)|$ tangent to C is the full symmetric group as C varies in $|-2K_\mathbb{P}|$.

Following the same line of argument as in [H], we show that G is the symmetric group by arguing that

1. G is twice transitive;
2. G contains a simple transposition.

To see that G is twice transitive, let W be the incidence correspondence

$$W = \big\{(C, I_1, I_2) : C \in |-2K_\mathbb{P}|,$$
$$\qquad I_1 \text{ and } I_2 \text{ are two different curves in } |\mathcal{O}_\mathbb{P}(F)| \text{ tangent to } C\big\}$$
$$\qquad \subset |-2K_\mathbb{P}| \times |\mathcal{O}_\mathbb{P}(F)| \times |\mathcal{O}_\mathbb{P}(F)|.$$

Saying that G is twice transitive is equivalent to saying that W is irreducible. To see that W is irreducible, it suffices to project W to $|\mathcal{O}_\mathbb{P}(F)| \times |\mathcal{O}_\mathbb{P}(F)|$. It is obvious that W dominates $|\mathcal{O}_\mathbb{P}(F)| \times |\mathcal{O}_\mathbb{P}(F)|$ and $p(W) = \{(I_1, I_2) : I_1 \neq I_2 \in |\mathcal{O}_\mathbb{P}(F)|\}$, where $p : W \to |\mathcal{O}_\mathbb{P}(F)| \times |\mathcal{O}_\mathbb{P}(F)|$ is the projection. And the fibers of $p : W \to |\mathcal{O}_\mathbb{P}(F)| \times |\mathcal{O}_\mathbb{P}(F)|$ are irreducible and have the same dimension everywhere. Therefore, W is irreducible and G is twice transitive.

To see that G contains a simple transposition, let C_0 be a curve in $|-2K_\mathbb{P}|$ having exactly one node p and smooth everywhere else. If there is a family of curves $C_t \in |-2K_\mathbb{P}|$ whose central fiber is C_0, two out of the 24 curves in $|\mathcal{O}_\mathbb{P}(F)|$ tangent to C_t will degenerate to the curve in $|\mathcal{O}_\mathbb{P}(F)|$ passing through the node p as $t \to 0$. It is easy to see that a loop around C_0 will be lifted to a simple transposition which transposes these two curves. $\qquad\square$

Let $\pi : \Delta \to \widetilde{\mathcal{Y}_g} \subset \widetilde{\mathcal{H}_g}$ be a morphism from the disk Δ to $\widetilde{\mathcal{Y}_g}$, where $\pi(0) = [R^s] \in \widetilde{\mathcal{R}_g^0}$ and $\pi(t)$ is a general point in $\mathcal{Y}_g$ for $t \in \Delta$ general. Let $\{p_1, p_2, ..., p_{16}\}$ be the vanishing locus of s.

Let $X = \widetilde{\mathcal{S}_g} \times_{\widetilde{\mathcal{H}_g}} \Delta$ be the family of K3 surfaces corresponding to π. As before, X has sixteen rational double points $p_1, p_2, ..., p_{16}$ lying on the double curve $E = R_1 \cap R_2$, where $X_0 = R = R_1 \cup R_2$.

Let r_{ij} be the points on E defined as before, i.e., $\mathcal{O}_E(2r_{ij}) = \mathcal{O}_E(F_i)$. Since $R \in \mathcal{R}_g^0$, $r_{1j} = r_{2j}$. Let q_i be the point on E such that $\mathcal{O}_E(p_i + q_i) = \mathcal{O}_E(F)$ for $i = 1, 2, ..., 16$.

Proposition 5.5. *As X_t degenerates to $X_0 = R$, the 24 rational curves in $|\mathcal{O}_{X_t}(F)|$ on X_t behave in the following way*

1. *two of them degenerate to $\overline{r_{1j}} \cup \overline{r_{2j}}$ for each $1 \leq j \leq 4$;*
2. *one of them degenerates to $\overline{p_i q_i}^{(1)} \cup \overline{p_i q_i}^{(2)}$ for each $1 \leq i \leq 16$.*

Proof. The first statement follows directly from the Caporaso-Harris theorem on tacnodes [CH, Lemma 4.1].

Let us resolve the sixteen double points $p_1, p_2, ..., p_{16}$ of X by blowing up R_1 at these points. Let $\widetilde{X}$ be the resulting family and $\widetilde{X}_0 = \widetilde{R_1} \cup R_2$, where $\widetilde{R_1}$ is the blowup of R_1 at the sixteen points. The total transform of $\overline{p_i q_i}^{(1)} \cup \overline{p_i q_i}^{(2)}$ consists of the proper transform I_i of $\overline{p_i q_i}^{(1)}$, the exceptional divisor P_i and $\overline{p_i q_i}^{(2)}$. Obviously, I_i and P_i meet transversely at a point w_i. By the same line of argument as in [C], we can deform $I_i \cup P_i \cup \overline{p_i q_i}^{(2)}$ to a curve on the general fiber while preserving a node in the neighborhood of w_i. This concludes the second statement. $\square$

Let $\Upsilon \subset X$ be a one-parameter family of curves whose general fiber $\Upsilon_t \in |\mathcal{O}_{X_t}(C + kF)|$ is irreducible and rational. Let $\Sigma_1, \Sigma_2, \Gamma_1, \Gamma_2$ be the components of Υ_0 defined as before.

Let $\overline{pq} = \overline{pq}^{(1)} \cup \overline{pq}^{(2)} \in |\mathcal{O}_R(F)|$ for $p, q \in E$ and $\mathcal{O}_E(p+q) = \mathcal{O}_E(F)$. If $\overline{pq} \subset \Upsilon_0$, from the proof of Proposition 2.2, we see that $\overline{pq}$ can only be one of the following

1. $p = p_i$ and $q = q_i$ for some i;
2. $p = q = r_{1j} = r_{2j}$ for some j; and if this is the case, r_{1j} does not lie on $\Gamma_1 \cup \Gamma_2$;
3. $p, q \notin \{r_{ij}, p_l, q_l\}$ and either p or q lies on $\Gamma_1 \cup \Gamma_2$.

The following two propositions say that we can be more specific in this case because Υ_0 is a limit of rational curves on elliptic K3's, while all the analysis we did in the proof of Proposition 2.2 is solely based on the fact that Υ_0 is a limit of rational curves on smooth K3 surfaces.

Proposition 5.6. *Let $\overline{pq} = \overline{pq}^{(1)} \cup \overline{pq}^{(2)} \in |\mathcal{O}_R(F)|$ and $\overline{pq} \subset \Upsilon_0$. If $\overline{pq} = \overline{p_i q_i}$ for some i, then*

1. *either $p = p_i \in \Gamma_1 \cup \Gamma_2$ or $q = q_i \in \Gamma_1 \cup \Gamma_2$;*
2. $\delta(\Upsilon_t, \overline{pq}) = \mathrm{Intsc}(\Sigma_1, E; p) + \mathrm{Intsc}(\Sigma_1, E; q)$.

Proposition 5.7. *We have $\overline{r_{1j}} \cup \overline{r_{2j}} \not\subset \Upsilon_0$ for any $1 \leq j \leq 4$.*

We need a few algebraic lemmas.

Let A be an integral domain. We use the notation $\overline{A}$ to denote the integral closure of A in the algebraic closure of its quotient field. It is well known that $\overline{\mathbb{C}[[t]]} = \mathbb{C}[[t, \sqrt{t}, \sqrt[3]{t}, ..., \sqrt[n]{t}, ...]]$.

Lemma 5.1. *Let $A = \mathbb{C}[x, [y]] \otimes_{\mathbb{C}} \overline{\mathbb{C}[[t]]}$ and $W \subset A$ be*

$$W = \left\{ \sum_{i=0}^{\infty} \left(a_i(t)y^i + b_i(t)xy^i \right) : a_i(t), b_i(t) \in \overline{\mathbb{C}[[t]]} \right\}.$$

Let $f(x, y, t) \in W$ satisfying $f(x, y, 0) \neq 0$. If $f(x, y, t)$ is reducible in A, then there exists $\alpha(t) \in \overline{\mathbb{C}[[t]]}$ such that $\alpha(0) = 0$ and $f(x, \alpha(t), t) = 0$.

Proof. Let $f(x, y, t) = g(x, y, t)h(x, y, t)$ where neither of $g(x, y, t)$ and $h(x, y, t)$ is a unit in A. Let $\deg_x g(x, y, t)$ and $\deg_x h(x, y, t)$ be the degrees of $g(x, y, t)$ and $h(x, y, t)$ as polynomials in x. Obviously, since $f(x, y, t) \in W$, at least one of $\deg_x g(x, y, t)$ and $\deg_x h(x, y, t)$ is zero. Assume that $\deg_x g(x, y, t) = 0$, i.e., $g(x, y, t) = r(y, t)$ for some $r(y, t) \in \mathbb{C}[[y]] \otimes_{\mathbb{C}} \overline{\mathbb{C}[[t]]}$. Since $g(x, y, t)$ is not a unit in A, $r(0, 0) = 0$. And $f(x, y, 0) \neq 0 \Rightarrow r(y, 0) \neq 0$. So using Weierstrass Preparation Theorem, we can show that there exists $\alpha(t) \in \overline{\mathbb{C}[[t]]}$ such that $\alpha(0) = 0$ and $r(\alpha(t), t) = 0$, which is what we want. $\square$

A corollary of Lemma 5.1 is

Corollary 5.1. *Let* $A = \mathbb{C}[x, y] \otimes_{\mathbb{C}} \overline{\mathbb{C}[[t]]}$ *and* $W \subset A$ *be*

$$W = \left\{ \sum \left(a_i(t)y^i + b_i(t)xy^i \right) : a_i(t), b_i(t) \in \overline{\mathbb{C}[[t]]} \right\}.$$

Let $f(x, y, t) \in W$ *satisfying* $f(x, y, 0) \neq 0$. *If* $f(x, y, t)$ *is irreducible in* A, *then* $f(x, y, t)$ *is also irreducible in* $\mathbb{C}[x, [y]] \otimes_{\mathbb{C}} \overline{\mathbb{C}[[t]]}$.

Proof. By Lemma 5.1, if $f(x, y, t)$ is reducible in $\mathbb{C}[x, [y]] \otimes_{\mathbb{C}} \overline{\mathbb{C}[[t]]}$, $f(x, \alpha(t), t) = 0$ for some $\alpha(t) \in \mathbb{C}[[t]]$ and $\alpha(0) = 0$. On the other hand, $f(x, y, t) = g(x, y, t)(y - \alpha(t)) + a(t)x + b(t)$ for some $g(x, y, t) \in A$ and $a(t), b(t) \in \overline{\mathbb{C}[[t]]}$. So we necessarily have $a(t) = b(t) = 0$ and $f(x, y, t)$ must be reducible in A. Contradiction. $\square$

Lemma 5.2. *Let* $C \subset \Delta^3_{xyz}/(xy = 0) = R_1 \cup R_2$ *be a reduced curve cut out by* $f(x, y, z) = 0$ *for some* $f(x, y, z) \in \mathbb{C}[[x, y, z]]$, *where* $R_1 = \Delta^3_{xyz}/(x = 0)$ *and* $R_2 = \Delta^3_{xyz}/(y = 0)$. *Suppose that* C *does not contain the double curve* $E = R_1 \cap R_2$. *Then*

$$(5.4) \qquad \delta(C) = \delta(C_1) + \delta(C_2) + C_1 \cdot E = \delta(C_1) + \delta(C_2) + C_2 \cdot E$$

where $C = C_1 \cup C_2$, $C_i \subset R_i$ *and the intersection* $C_i \cdot E$ *is taken on* R_i *for* $i = 1, 2$.

Proof. Let $\widetilde{\mathcal{O}_C}$ and $\widetilde{\mathcal{O}_{C_i}}$ be the normalizations of the coordinate rings of C and C_i, respectively. By the definition of δ-invariants, $\delta(C) = l(\widetilde{\mathcal{O}_C}/\mathcal{O}_C)$, where $l(M)$ is the length of an $\mathcal{O}_C$ module M.

Obviously, $\widetilde{\mathcal{O}_C} = \widetilde{\mathcal{O}_{C_1}} \times \widetilde{\mathcal{O}_{C_2}}$. Therefore,

$$\delta(C) = l(\widetilde{\mathcal{O}_C}/\mathcal{O}_C) = l\left((\widetilde{\mathcal{O}_{C_1}} \times \widetilde{\mathcal{O}_{C_2}})/\mathcal{O}_C \right)$$

$$= l\left((\widetilde{\mathcal{O}_{C_1}} \times \widetilde{\mathcal{O}_{C_2}})/(\mathcal{O}_{C_1} \times \mathcal{O}_{C_2})\right) + l((\mathcal{O}_{C_1} \times \mathcal{O}_{C_2})/\mathcal{O}_C)$$

$$= l(\widetilde{\mathcal{O}_{C_1}}/\mathcal{O}_{C_1}) + l(\widetilde{\mathcal{O}_{C_2}}/\mathcal{O}_{C_2}) + C_i \cdot E,$$

for $i = 1, 2$. $\qquad\square$

Lemma 5.3. *Let $p, q, f \in \mathbb{C}[z]$ be the polynomials in z such that $\deg f \leq 1$, $\gcd(p, q) = 1$ and $(p^2 - fq^2)^2 - q^4$ is a perfect square in $\mathbb{C}[z]$, where $\gcd(p, q)$ is the greatest common divisor of p and q. Then*

1. *both p and q are constants;*
2. *f is a constant if $q \neq 0$.*

Proof. Let $f_1 = f - 1$ and $f_2 = f + 1$. Obviously, $\gcd(p^2 - f_1q^2, p^2 - f_2q^2) = 1$ since $\gcd(p, q) = 1$. And since $(p^2 - f_1q^2)(p^2 - f_2q^2)$ is a perfect square, both $p^2 - f_1q^2$ and $p^2 - f_2q^2$ are perfect squares.

It is easy to show that if one of p and q is a constant, the other is a constant too. On the other hand, if both are constants and $q \neq 0$, f has to be a constant too. Suppose that $p, q \notin \mathbb{C}$.

Solve $p^2 - f_iq^2 = r_i^2$ for $i = 1, 2$ and we obtain that there exist $q_1, q_2, q_3, q_4 \in \mathbb{C}[z]$ such that $q = q_1q_2 = q_3q_4$ and

$$p = \frac{q_1^2 + f_1q_2^2}{2} = \frac{q_3^2 + f_1q_4^2}{2}.$$

Since $q_1q_2 = q_3q_4$, there exist $s, t, u, v \in \mathbb{C}[z]$ such that $q_1 = su$, $q_2 = tv$, $q_3 = tu$ and $q_4 = sv$. Therefore,

$$p = \frac{s^2u^2 + f_1t^2v^2}{2} = \frac{t^2u^2 + f_2s^2v^2}{2}$$

and hence

$$\frac{u^2 - f_1v^2}{u^2 - f_2v^2} = \frac{s^2}{t^2}$$

which implies that $(u^2 - f_1v^2)(u^2 - f_2v^2)$ is a perfect square.

Obviously, $\gcd(u, v) = 1$. Since $q \notin \mathbb{C}$, u and v cannot both be constants. Combining with our previous argument, neither of u and v is a constant. Therefore, $\max(\deg u, \deg v) < \deg q \leq \max(\deg p, \deg q)$. So this procedure cannot go on forever. A contradiction. $\qquad\square$

Now let us go back to the proof of Proposition 5.6 and 5.7.

Our proofs of both statements are based on the construction a blowup sequence over $\overline{pq}$ and an induction on the multiplicity of $\overline{pq}$ in Υ_0. It turns out that this process can be described more clearly if we study the behaviors of X and Υ in the analytic neighborhood of $\overline{pq}$, or alternatively, study the formal completion of X and Υ along $\overline{pq}$. But we will stick to the language of analytic geometry for it being more intuitive.

The following proposition is lengthy to state due to the fact that we need to give a precise description of X and Υ in the neighborhood of $\overline{pq}$. But such description is necessary for the purpose of induction.

Proposition 5.8. *Let X be a flat family of analytic surfaces over disk Δ_t whose central fiber $X_0 = R_1 \cup R_2$ where $R_i \cong \Delta \times \mathbb{P}^1$ for $i = 1, 2$. Suppose that $R_1 \cap R_2 = \Delta(p) \cup \Delta(q)$ where $\Delta(p)$ and $\Delta(q)$ are disks centering at points p and q, respectively, and $\Delta(p)$ and $\Delta(q)$ are closed subschemes of R_i $(i = 1, 2)$. Suppose that X is locally given by $\operatorname{Spec} \mathbb{C}[[x, y, z, t]]/(xy - t^a z^b)$ and $\operatorname{Spec} \mathbb{C}[[x, y, z, t]]/(xy - t^a)$ at p and q, respectively, where $a, b \in \mathbb{Z}$, $a > 0$ and $b = 0$ or 1.*

Let $z \in \Gamma(\mathcal{O}_X)$ and let $z = 0$ cut out the "banana" curve $\overline{pq} = \overline{pq}^{(1)} \cup \overline{pq}^{(2)}$ on X_0, where $\overline{pq}^{(i)} \cong \mathbb{P}^1$, $\overline{pq}^{(i)} \subset R_i$ and each $\overline{pq}^{(i)}$ meets $\Delta(p)$ and $\Delta(q)$ transversely at p and q, respectively, for $i = 1, 2$.

Let $\mathcal{O}_X(C)$ be a line bundle on X such that the restrictions of $\mathcal{O}_X(C)$ to $\overline{pq}^{(i)}$ are $\mathcal{O}_{\mathbb{P}^1}(1)$ and let s_1 and s_2 be two global sections of $\mathcal{O}_X(C)$ which generate $H^0(\mathcal{O}_{\mathbb{P}^1}(1))$ when restricted to $\overline{pq}^{(i)}$ for $i = 1, 2$.

Let $f(s_1, s_2, z, t) \in \mathbb{C}[s_1, s_2, [z, t]]$ lie in

$$f(s_1, s_2, z, t) \in \left\{ \sum_{i=0}^{\infty} \left(a_i(t)s_1 z^i + b_i(t)s_2 z^i \right) : a_i(t), b_i(t) \in \mathbb{C}[[t]] \right\}$$

and $f(s_1, s_2, z, 0) \neq 0$. Suppose that $f(s_1, s_2, z, t)$ is irreducible in $\mathbb{C}[s_1, s_2, [z]] \otimes_{\mathbb{C}} \overline{\mathbb{C}[[t]]}$.

Let $\Upsilon \subset X$ be the subscheme of X cut out by $f(s_1, s_2, z, t) = 0$. Obviously, Υ_0 consists of a multiple of $\overline{pq}$ and two disks $\Gamma_1 \subset R_1$ and $\Gamma_2 \subset R_2$. Let μ be the multiplicity of $\overline{pq}$ in Υ_0. Suppose that $\mu > 0$.

Let $\widetilde{\Upsilon}$ be the nodal reduction of Υ. Then

1. *$\widetilde{\Upsilon}_0$ has at most two connected components with Γ_1 and Γ_2 in each component, respectively;*
2. *$\widetilde{\Upsilon}_0$ is connected if $b = 0$ or $p, q \notin \Gamma_i$ for $i = 1, 2$;*
3. *if $\widetilde{\Upsilon}_0$ has two connected components and $\Gamma_1 \neq \Gamma_2$, then*

(5.5) $$\delta(\Upsilon_t) = 2\mu + \Gamma_1 \cdot \Delta(p) + \Gamma_1 \cdot \Delta(q),$$

   where the intersections are taken on R_1 (let $\Gamma_1 \cdot \Delta(p) = 0$ if $p \notin \Gamma_1$ and $\Gamma_1 \cdot \Delta(q) = 0$ if $q \notin \Gamma_1$).

Proof. Since $\overline{pq}^{(i)}$ $(i = 1, 2)$ meet $\Delta(p)$ transversely, we can choose the local coordinates (x, y, z) of X at p such that

$$X \cong \Delta^4_{xyzt}/\left(xy = t^a(z^b + \alpha(t))\right)$$

at p and the local function z is exactly the restriction of the global z as defined in the proposition, where $\alpha(t) \in \mathbb{C}[[t]]$ and $\alpha(0) = 0$. Of

course, if $b = 0$, we can make $\alpha(t)$ vanish. If $b = 1$, we can replace the global function z by $z - \alpha(t)$, which we will call a translation on z. So eventually, we arrive at

$$(5.6) \qquad X \cong \Delta^4_{xyzt}/\left(xy = t^a z^b\right)$$

at p and the local coordinates (x, y, z) are chosen such that the local function z is the restriction of the global function z.

We may put the defining equation of Υ in the following form

$$
(5.7) \qquad
\begin{aligned}
f(w, z, t) &= w \prod_{i=1}^{\mu} \left(z + a_i(t) \right) \\
&\quad + t^\beta u(z, t) \prod_{j=1}^{m} \left(z + b_j(t) \right) = 0
\end{aligned}
$$

where $w = (c_{11}s_1 + c_{12}s_2)/(c_{21}s_1 + c_{22}s_2)$ for some

$$\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C}[[z, t]]),$$

$u(z, t) \in \mathbb{C}[[z, t]]$, $u(0, 0) \neq 0$, $a_i(t), b_j(t) \in \overline{\mathbb{C}[[t]]}$, $a_i(0) = b_j(0) = 0$, $\beta > 0$ and $\mu > m$. And we arrange $\{a_i(t)\}$ and $\{b_j(t)\}$ in the order that

$$(5.8) \qquad \nu(a_1(t)) \leq \nu(a_2(t)) \leq ... \leq \nu(a_\mu(t))$$

and

$$(5.9) \qquad \nu(b_1(t)) \leq \nu(b_2(t)) \leq ... \leq \nu(b_m(t)),$$

where $\nu(c(t))$ is the valuation of $c(t) \in \overline{\mathbb{C}[[t]]}$ and we let $\nu(0) = \infty$. If $b = 0$, we may further do a translation on z and assume that

$$(5.10) \qquad \sum_{i=1}^{\mu} a_i(t) = 0.$$

It is obvious that $f(w, z, t)$ is irreducible in $\mathbb{C}[w, [z]] \times_{\mathbb{C}} \overline{\mathbb{C}[[t]]}$ since $f(s_1, s_2, z, t)$ is irreducible in $\mathbb{C}[s_1, s_2, [z]] \otimes_{\mathbb{C}} \overline{\mathbb{C}[[t]]}$.

We want to blow up X along a subscheme cut out by $z = t^\gamma = 0$ for some γ. Let

$$(5.11) \qquad \gamma = \min \left\{ \nu(a_1(t)), \frac{\beta}{\mu - m}, \gamma_0 \right\}$$

where

$$\gamma_0 = \min_{1 \leq k \leq m} \frac{1}{\mu - m + k} \left(\beta + \sum_{j=1}^{k} \nu(b_j(t)) \right).$$

Alternatively, we may define γ as the largest number such that

$$t^{\mu\gamma}\big|f(w, t^\gamma z, t).$$

After a base change, we may assume that $\gamma \in \mathbb{Z}$.

Let $\widetilde{X}$ be the blowup of X along the subscheme cut out by $z = t^\gamma = 0$ and let $\Upsilon' \subset \widetilde{X}$ be the proper transform of Υ. The central fiber $\widetilde{X}_0$ consists of four surfaces $R_1 \cup R_2 \cup (\overline{pq}^{(1)} \times \mathbb{P}^1) \cup (\overline{pq}^{(2)} \times \mathbb{P}^1)$, where $R_i \cap (\overline{pq}^{(i)} \times \mathbb{P}^1) = \overline{pq}^{(i)}$ $(i = 1, 2)$ and $(\overline{pq}^{(1)} \times \mathbb{P}^1) \cap (\overline{pq}^{(2)} \times \mathbb{P}^1) = (\{p\} \times \mathbb{P}^1) \cup (\{q\} \times \mathbb{P}^1)$.

By (5.7), Υ' is cut out on $\widetilde{X}$ by

$$(5.12) \qquad\qquad\qquad f_1(w, z_1, t) = 0$$

where $z_1 = z/t^\gamma$ and

$$(5.13) \qquad\qquad f_1(w, z_1, t) = t^{-\mu\gamma} f(w, t^\gamma z_1, t).$$

Let

$$(5.14) \qquad f_1(w, z, 0) = \left(wh_1(z_1) + h_2(z_1)\right) \prod_{i=1}^{l} (z_1 + \alpha_i)^{\mu_i}$$

with $z_1 = z/t^\gamma$, where $h_1(z_1), h_2(z_1) \in \mathbb{C}[z_1]$, $\deg h_1(z_1) > \deg h_2(z_1)$, $wh_1(z_1) + h_2(z_1)$ is irreducible in $\mathbb{C}[w, z_1]$, $\alpha_1, \alpha_2, ..., \alpha_l \in \mathbb{C}$ are l distinct numbers, $\mu_i \in \mathbb{Z}$, $\mu_i > 0$ and $\deg h_1(z_1) + \sum_{i=1}^{l} \mu_i = \mu$.

By (5.14), we see that the central fiber Υ'_0 of Υ' consists of components $\Gamma_1, \Gamma_2, I_0, I_1, ..., I_l$ where

1. each I_j has two components $I_j^{(1)}$ and $I_j^{(2)}$ with $I_j^{(k)} \subset \overline{pq}^{(k)} \times \mathbb{P}^1$ for $j = 0, 1, ..., l$ and $k = 1, 2$;
2. $I_0^{(k)}$ is cut out by $wh_1(z_1) + h_2(z_1) = 0$ on $\overline{pq}^{(k)} \times \mathbb{P}^1$; $I_j^{(k)}$ is cut out by $z_1 + \alpha_j = 0$ on $\overline{pq}^{(k)} \times \mathbb{P}^1$ for $j = 1, 2, ..., l$ and $k = 1, 2$ (here we regard (w, z) as the affine coordinates of $\overline{pq}^{(k)} \times \mathbb{P}^1 \cong \mathbb{P}^1 \times \mathbb{P}^1$);
3. $I_0^{(k)}$ projects to $\overline{pq}^{(k)}$ with a degree $\mu_0 = \deg h_1(z_1)$ map for $k = 1, 2$ (if $\mu_0 = 0$, $I_0^{(k)}$ contracts to the point $\Gamma_k \cap \overline{pq}^{(k)}$);
4. $I_j^{(k)}$ has multiplicity μ_j in Υ'_0; $I_j^{(1)}$ and $I_j^{(2)}$ meet at two points $p_j \in \{p\} \times \mathbb{P}^1$ and $q_j \in \{q\} \times \mathbb{P}^1$ for $j = 1, 2, ..., l$;
5. $I_1, I_2, ..., I_l$ are disjoint from each other; $I_0^{(k)}$ meets each $I_j^{(k)}$ at exactly one point for $j = 1, 2, ..., l$ and $k = 1, 2$;
6. $I_0^{(k)}$ meets Γ_k at a point $r_k \in \overline{pq}^{(k)} \times \mathbb{P}^1$ with coordinates $w = 1/z_1 = 0$ and $r_k \notin I_j$ for $j = 1, 2, ..., l$ and $k = 1, 2$.

By Lemma 2.1, the way in which $I_0^{(k)}$ are connected to Γ_k $(k = 1, 2)$ on $\widetilde{\Upsilon}_0$ can be described as follows.

($*$) Either $I_0^{(k)}$ and Γ_k are joined by curves contracting to r_k on $\widetilde{\Upsilon}_0$ or $I_0^{(k)}$ and Γ_{3-k} are joined by curves contracting to r_k on $\widetilde{\Upsilon}_0$ for $k = 1, 2$ (the latter could happen when $r_1 = r_2 \in I_0^{(1)} \cap I_0^{(2)}$).

We will argue by induction on the pair (μ, b). We define $(\mu, b) < (\mu', b')$ if $\mu < \mu'$ or $\mu = \mu'$ and $b < b'$.

$\underline{l > 0}$

Take a component I_j and an analytic neighborhood U of $\widetilde{X}$ around I_j. Let $Y = U \cap \Upsilon'$. Then Y and U have all the properties described in the proposition. For example, Y is cut out on U by $f_2(w, z_2, t) = 0$ where

$$(5.15) \qquad f_2(w, z_2, t) = f_1(w, z_2 - \alpha_j, t) = t^{-\mu\gamma} f(w, t^\gamma(z_2 - \alpha_j), t)$$

By Lemma 5.1, $f_2(w, z_2, t)$ is irreducible in $\mathbb{C}[w, [z_2]] \otimes_{\mathbb{C}} \overline{\mathbb{C}[[t]]}$ since $f(w, z, t)$ is irreducible in $\mathbb{C}[w, [z]] \otimes_{\mathbb{C}} \overline{\mathbb{C}[[t]]}$. And in the neighborhoods of p_j and q_j, U is given by $\operatorname{Spec} \mathbb{C}[[x, y, z, t]]/(xy - t^{(a+\gamma)} z^c)$ and $\operatorname{Spec} \mathbb{C}[[x, y, z, t]]/(xy - t^{(a+\gamma)})$, respectively, where $c = 0$ if either $b = 0$ or $\alpha_j \neq 0$ and $c = 1$ otherwise. The central fiber Y_0 of Y contains I_j with multiplicity μ_j plus two disks Γ_1' and Γ_2' attached to $I_j^{(1)}$ and $I_j^{(2)}$, respectively. Obviously, Γ_k' is a piece of $I_0^{(k)}$ for $k = 1, 2$.

To apply the induction hypothesis, we have to check that $(\mu_j, c) < (\mu, b)$, which is an easy consequence of (5.10) and the way we choose the number γ.

Let $\widetilde{Y}$ be the nodal reduction of Y. By the induction hypothesis, $\widetilde{Y}_0$ has at most two connected components with Γ_1' and Γ_2' in each component. So combining with ($*$), we see that $\widetilde{\Upsilon}_0$ has at most two connected components with Γ_1 and Γ_2 in each component. Furthermore, as long as there exists one I_j such that the corresponding $\widetilde{Y}_0$ is connected, $\widetilde{\Upsilon}_0$ is connected. By the induction hypothesis, $\widetilde{Y}_0$ is connected if $c = 0$ or I_0 meets I_j at points other than q_j for $1 \leq j \leq l$. Therefore, $\widetilde{\Upsilon}_0$ is connected if one of the following holds:

1. $b = 0$;
2. $\alpha_j \neq 0$ for some $1 \leq j \leq l$;
3. I_0 meets I_j at points other than p_j and q_j for some $1 \leq j \leq l$.

Let us deal with the remaining case that $b = l = 1$, $\alpha_1 = 0$ and I_0 meets I_1 at p_1 or q_1. We necessarily have $\mu_0 > 0$ due to the way we choose the number γ. So $I_0^{(1)}$ and $I_0^{(2)}$ should meet at (at least) one point p_0 on $\{p\} \times \mathbb{P}^1$ and one point q_0 on $\{q\} \times \mathbb{P}^1$.

If $\{p_0, q_0\} \not\subset \{r_1, r_2, p_1, q_1\}$ for some $p_0 \in I_0^{(k)} \cap (\{p\} \times \mathbb{P}^1)$ and $q_0 \in I_0^{(k)} \cap (\{q\} \times \mathbb{P}^1)$, it is not hard to see by Lemma 2.1 that $I_0^{(1)}$ and $I_0^{(2)}$ are joined by a chain of curves contracting to p_0 or q_0 (and hence to p and q) on $\widetilde{\Upsilon}_0$ and $\widetilde{\Upsilon}_0$ is hence connected.

If $\{p_0, q_0\} \subset \{r_1, r_2, p_1, q_1\}$ for any $p_0 \in I_0^{(k)} \cap (\{p\} \times \mathbb{P}^1)$ and $q_0 \in I_0^{(k)} \cap (\{q\} \times \mathbb{P}^1)$, then $I_0^{(1)}$ and $I_0^{(2)}$ only meet at p_0 and q_0 and

A. $p \in \Gamma_i$ $(i = 1, 2)$, $r_1 = r_2 = p_0$ and $q_0 = q_1$; OR
B. $q \in \Gamma_i$ $(i = 1, 2)$, $r_1 = r_2 = q_0$ and $p_0 = p_1$.

We need to prove (5.5) in both cases if we further assume that $\widetilde{\Upsilon}_0$ has two connected components and $\Gamma_1 \neq \Gamma_2$. Let us work on case A and case B will follow from the same argument.

By Lemma 5.2 and the induction hypothesis,

$$
\begin{aligned}
\delta(\Upsilon_t) = \delta(\Upsilon_t') &= \delta(\Upsilon_t', p_0) + \delta(\Upsilon_t', I_1) \\
&= (\mu_0 + \Gamma_1 \cdot \Delta(p)) + (\mu_0 + 2\mu_1) \\
&= 2(\mu_0 + \mu_1) + \Gamma_1 \cdot \Delta(p) = 2\mu + \Gamma_1 \cdot \Delta(p).
\end{aligned}
$$

$\underline{l = 0}$

Since $\mu_0 = \mu > 0$, $I_0^{(1)}$ and $I_0^{(2)}$ meet at (at least) one point $p_0 \in \{p\} \times \mathbb{P}^1$ and one point $q_0 \in \{q\} \times \mathbb{P}^1$.

If $b = 0$ or $q_0 \notin \{r_1, r_2\}$ for some $q_0 \in I_0^{(k)} \cap (\{q\} \times \mathbb{P}^1)$, then by Lemma 2.1, $I_0^{(1)}$ and $I_0^{(2)}$ are joined by a chain of curves contracting to either p_0 or q_0 on $\widetilde{\Upsilon}_0$ and $\widetilde{\Upsilon}_0$ is hence connected.

If $b = 1$, $q_0 = r_1 = r_2$ and $I_0^{(k)}$ meets $\{p\} \times \mathbb{P}^1$ at (at least) two different points p_0 and p_0', then by Lemma 2.1, $I_0^{(1)}$ and $I_0^{(2)}$ are joined by a chain of curves contracting to either p_0 or p_0' on $\widetilde{\Upsilon}_0$ and $\widetilde{\Upsilon}_0$ is hence connected.

If $b = 1$, $q_0 = r_1 = r_2$, $I_0^{(k)}$ meets $\{p\} \times \mathbb{P}^1$ only at one point p_0 and $\mu \geq 2$, then it follows from Corollary 2.1 that $I_0^{(1)}$ and $I_0^{(2)}$ are joined by a chain of curves contracting to p_0 on $\widetilde{\Upsilon}_0$ and $\widetilde{\Upsilon}_0$ is hence connected.

So the only case left is that $b = 1$, $q_0 = r_1 = r_2$ and $\mu = 1$. Obviously, it follows from $(*)$ that $\widetilde{\Upsilon}_0$ has at most two connected components with Γ_1 and Γ_2 in each component, respectively. We need to verify (5.5) if we further assume that $\widetilde{\Upsilon}_0$ has two connected components and $\Gamma_1 \neq \Gamma_2$.

By Lemma 5.2,

$$
\begin{aligned}
\delta(\Upsilon_t) &= \delta(\Upsilon_t, p) + \delta(\Upsilon_t, q) \\
&= 1 + (1 + \Gamma_1 \cdot \Delta(q)) = 2\mu + \Gamma_1 \cdot \Delta(q).
\end{aligned}
$$

$\square$

Proposition 5.6 follows more or less directly from Proposition 5.8.

Proof of Proposition 5.6. Suppose that $H^0(\mathcal{O}_X(C + kF))$ is generated by $g + 1$ global sections $Y_0, Y_1, ..., Y_g$ as a free $\mathbb{C}[[t]]$ module. Let $H^0(\mathcal{O}_X(F))$ be generated by two global sections Z_0 and Z_1 where $Z_1 = 0$ cuts out $\overline{pq}$ on X_0, Let $W_1, W_2 \in H^0(\mathcal{O}_X(F))$ be two global sections of $\mathcal{O}_X(C + F)$ whose restrictions to $\overline{pq}^{(i)}$ generate $H^0(\mathcal{O}_{\mathbb{P}^1}(1))$. All these $Y_0, Y_1, ..., Y_g, Z_0, Z_1, W_1, W_2$ exist after a base change.

Let $s_1 = W_1/Z_0$ and $s_2 = W_2/Z_0$. When restricted to an analytic neighborhood of $\overline{pq}$, s_1 and s_2 are holomorphic sections of $\mathcal{O}_X(C)$ and

$$\left\{ \sum_{i=0}^{g} a_i(t)Y_i : a_i(t) \in \mathbb{C}[[t]] \right\}$$
$$\subset \left\{ \sum_{i=0}^{k} (b_i(t)s_1 Z_0^i Z_1^{k-i} + c_i(t)s_2 Z_0^i Z_1^{k-i}) : b_i(t), c_i(t) \in \mathbb{C}[[t]] \right\}.$$

Therefore, Υ is locally cut out by $f(s_1, s_2, z, t) = 0$ as described in Proposition 5.8, where $z = Z_1/Z_0$. Since Υ_t is (geometrically) irreducible, it follows from Lemma 5.1 that $f(s_1, s_2, z, t)$ is irreducible in $\mathbb{C}[s_1, s_2, [z]] \otimes_{\mathbb{C}} \overline{\mathbb{C}[[t]]}$. Then by Proposition 5.8, Γ_1 and Γ_2 must pass through either p or q; otherwise, Γ_1 and Γ_2 will be joined by a chain of curves on $\widetilde{\Upsilon}_0$ whose images lie in $\overline{pq}$ and we have shown in the proof of Proposition 2.2 that Γ_1 and Γ_2 are joined by a chain of curves somewhere else on $\widetilde{\Upsilon}_0$, which leads to a contradiction.

If Γ_1 and Γ_2 pass through p or q, then by Proposition 5.8,

$$\delta(\Upsilon_t, \overline{pq}) = 2\mu + \text{Intsc}(\Gamma_1, E; p) + \text{Intsc}(\Gamma_1, E; q)$$
$$= \text{Intsc}(\Sigma_1, E; p) + \text{Intsc}(\Sigma_1, E; q)$$

where μ is the multiplicity of $\overline{pq}$ in Υ_0. $\qquad \square$

Proposition 5.9. *Let $X \subset \Delta_{xyzt}^4$ be defined by*

(5.16) $$y(y + x^2 + t^a z^b) = \lambda t^c$$

where $a, b, c \in \mathbb{Z}$, $a + b > 0$, $b = 0$ or 1, $c > 0$, $\lambda = \lambda(x, y, z, t) \in \mathbb{C}[[x, y, z, t]]$ and $\lambda(0, 0, 0, 0) \neq 0$.
Let $f(w, z, t) \in \mathbb{C}[w, [z, t]]$ lie in

$$f(w, z, t) \in \left\{ \sum_{i=0}^{\infty} \left(a_i(t)z^i + b_i(t)wz^i \right) : a_i(t), b_i(t) \in \mathbb{C}[[t]] \right\}$$

and $f(w, z, 0) = wz^{\mu}$ for some $\mu \in \mathbb{Z}$ and $\mu \geq 0$. Suppose that $f(w, z, t)$ is irreducible in $\mathbb{C}[w, [z]] \otimes_{\mathbb{C}} \overline{\mathbb{C}[[t]]}$.

Let $\Upsilon \subset X$ be a flat family of curves cut out by $f(w, z, t) = 0$ on X, where $w = w(x, y, z, t) \in \mathbb{C}[[x, y, z, t]]$ satisfying

$$w(0, 0, 0, 0) = 0 \ and \ \frac{\partial w}{\partial x}(0, 0, 0, 0) \neq 0.$$

Let $I_1, I_2, \Gamma_1, \Gamma_2$ be the irreducible components of Υ_0 where $I = I_1 \cup I_2$ is cut out by $z = 0$ and $\Gamma_1 \cup \Gamma_2$ is cut out by $w = 0$ on $X_0 = R_1 \cup R_2$. Let $\Gamma_j \subset R_j$ and $I_j \subset R_j$ for $j = 1, 2$. Suppose that Γ_j meets E properly for $j = 1, 2$. Let $\widetilde{\Upsilon}$ be the nodal reduction of Υ. Then either Γ_1 and Γ_2 lie on the same connected component of $\widetilde{\Upsilon}_0$ or the dual graph of $\widetilde{\Upsilon}_0$ contains a circuit.

Proof. If $a = 0$, the conclusion is more or less obvious. Notice that $b = 1$ if $a = 0$. Hence Γ_j meets E transversely, while I_j is tangent to E with multiplicity 2. So it follows from Lemma 2.1 that Γ_1 and Γ_2 lie on the same connected component of $\widetilde{\Upsilon}_0$.

It is also trivial if $\mu = 0$.

Suppose that $a > 0$ and $\mu > 0$. Again, we will argue by induction on (μ, b). We need to blow up Υ three times along subschemes on the central fiber in order to "bring down" the pair (μ, b).

We may put $f(w, z, t) = 0$ in the form (5.7) and also assume $\{a_i(t)\}$ and $\{b_j(t)\}$ to satisfy (5.8) and (5.9) and to satisfy (5.10) if $b = 0$.

Since Γ_i meets E properly, $w(0, 0, z, 0) \neq 0$. Let

$$w(0, 0, z, t) = \sum_{i=1}^{\eta}(z + c_i(t))$$

where $\eta > 0$, $c_i(t) \in \overline{\mathbb{C}[[t]]}$ and $c_i(0) = 0$. We arrange $\{c_i(t)\}$ in the order that

$$\nu(c_1(t)) \leq \nu(c_2(t)) \leq \ldots \leq \nu(c_\eta(t))$$

Let

$$(5.17) \quad \gamma = \min\left\{\frac{a}{2\eta - b}, \ \frac{c}{4\eta}, \ \nu(c_1(t)), \ \nu(a_1(t)), \ \frac{\beta}{\mu - m + \eta}, \ \gamma_0\right\},$$

where

$$\gamma_0 = \min_{1 \leq k \leq m} \frac{1}{\mu - m + k + \eta}\left(\beta + \sum_{j=1}^{k}\nu(b_j(t))\right).$$

Alternatively, γ can be defined as the largest number such that

$$t^{(\mu+\eta)\gamma}\big| f\big(w(t^{\eta\gamma}x, t^{2\eta\gamma}y, t^\gamma z, t), t^\gamma z, t\big) \text{ and } t^{4\eta\gamma}\big| g(t^{\eta\gamma}x, t^{2\eta\gamma}y, t^\gamma z, t),$$

where we let $g(x, y, z, t) = y(y + x^2 + t^a z^b) - \lambda t^c$.

Let $\gamma \in \mathbb{Z}$ after a base change. Let Υ' be the blowup of Υ at the 0-dimensional subscheme $x = y = z = t^\gamma = 0$. Then Υ' is given by

$$(5.18) \qquad y_1 \left(y_1 + t^\gamma x_1^2 + t^{a-(1-b)\gamma} z_1^b \right) = \lambda t^{c-2\gamma}$$

and

$$(5.19) \qquad\qquad\qquad f_1(w_1, z_1, t) = 0$$

where $x_1 = x/t^\gamma$, $y_1 = y/t^\gamma$, $z_1 = z/t^\gamma$,

$$w_1 = w_1(x_1, y_1, z_1, t) = t^{-\gamma} w(t^\gamma x_1, t^\gamma y_1, t^\gamma z_1, t)$$

and

$$f_1(w_1, z_1, t) = t^{-(\mu+1)\gamma} f(t^\gamma w_1, t^\gamma z_1, t).$$

By (5.18) and (5.19), we can describe the central fiber Υ'_0 of Υ as follows.

The exceptional locus of $\Upsilon' \to \Upsilon$ is a reducible and nonreduced curve F cut out on $\mathbb{P}^3$ by

$$(5.20) \qquad \begin{cases} y_1^2 = 0 \\ f_1\left(w_1(x_1, y_1, z_1, 0), z_1, 0\right) = 0 \end{cases}$$

where $x_1 = X_1/T_1$, $y_1 = Y_1/T_1$ and $z_1 = Z_1/T_1$ are the affine coordinates of $\mathbb{P}^3$ with corresponding homogeneous coordinates (X_1, Y_1, Z_1, T_1).

It is not hard to see that $\Gamma_1 \cup \Gamma_2$ meets F at the point p with coordinates

$$\left(-\frac{\partial w}{\partial z}(0,0,0,0), 0, \frac{\partial w}{\partial x}(0,0,0,0), 0 \right)$$

and $I_1 \cup I_2$ meets F at the point q with coordinates $(1, 0, 0, 0)$. Obviously, $p \neq q$, i.e., the blowup $\Upsilon' \to \Upsilon$ has separated $\Gamma_1 \cup \Gamma_2$ from $I_1 \cup I_2$.

The point $p = \Gamma_k \cap F$ lies on a unique irreducible component Σ of F and Σ has multiplicity 2 in F.

We may continue to use $\widetilde{\Upsilon}$ to denote the nodal reduction of Υ'. By Lemma 2.1, each Γ_i is joined to a component dominating Σ on $\widetilde{\Upsilon}_0$ by a chain of curves contracting to the point p for $i = 1, 2$. Therefore, in order to show that Γ_1 and Γ_2 lie on the same connected component of $\widetilde{\Upsilon}_0$, it suffices to show that

($*$) all the components dominating Σ lie on the same connected component of $\widetilde{\Upsilon}_0$.

This line of argument naturally leads to the second and the third blowups. This time we need to blow up Υ' along some subscheme supported along F_{red}.

Let Υ'' be the blowup of Υ' along the subscheme cut out by $x_1 = t^{(\eta-1)\gamma} = 0$. And let $\mathcal{Y}$ be the blowup of Υ'' along the subscheme cut out by $y_1 = t^{(2\eta-1)\gamma} = 0$. Notice that if $\eta = 1$, we do not need the intermediate family Υ''. Finally, we obtain $\mathcal{Y}$, which is given by

$$(5.21) \qquad y_2\left(y_2 + x_2^2 + t^{a-(2\eta-b)\gamma}z_1^b\right) = \lambda t^{c-4\eta\gamma}$$

and

$$(5.22) \qquad\qquad\qquad f_2(w_2, z_1, t) = 0$$

where $x_2 = x_1/t^{(\eta-1)\gamma}$, $y_2 = y_1/t^{(2\eta-1)\gamma}$,

$$w_2 = w_2(x_2, y_2, z_1, t) = \frac{1}{t^{(\eta-1)\gamma}}w_1(t^{(\eta-1)\gamma}x_2, t^{(2\eta-1)\gamma}y_2, z_1, t)$$

and

$$f_2(w_2, z_1, t) = \frac{1}{t^{(\eta-1)\gamma}}f_1(t^{(\eta-1)\gamma}w_2, z_1, t).$$

Let

$$(5.23) \qquad f_2(w_2, z_1, 0) = \left(w_2 h_1(z_1) + h_2(z_1)\right)\prod_{i=1}^{l}(z_1 + \alpha_i)^{\mu_i} = 0$$

where $h_1(z_1), h_2(z_1) \in \mathbb{C}[z_1]$, $\deg h_1(z_1) > \deg h_2(z_1)$, $w_2 h_1(z_1) + h_2(z_1)$ is irreducible in $\mathbb{C}[w_2, z_1]$, $\alpha_1, \alpha_2, ..., \alpha_l \in \mathbb{C}$ are l distinct numbers, $\mu_i \in \mathbb{Z}$, $\mu_i > 0$ and $\deg h_1(z_1) + \sum_{i=1}^{l}\mu_i = \mu$.

By (5.23), there are curves $\widetilde{\Sigma} \subset \mathcal{Y}_0$ and $J_i \subset \mathcal{Y}_0$ where $\widetilde{\Sigma}$ is cut out by $w_2 h_1(z_1) + h_2(z_1) = 0$ and J_i is cut out by $z_1 + \alpha_i = 0$ for $i = 1, 2, ..., l$. Obviously, $\widetilde{\Sigma}$ dominates Σ with a degree two map.

To be precise, $\widetilde{\Sigma}$ and J_i are complete curves lying on the surface $\mathbb{P}\left(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)\right)$. However, to argue $(*)$, we only need to study their affine parts. So we will treat them as affine curves in $\mathbb{A}^3 = \operatorname{Spec}\mathbb{C}[x_2, y_2, z_1]$. For example, when we talk about the intersections between these curves, we are talking about their intersections in $\mathbb{A}^3$.

<u>$c = 4\eta\gamma$</u>

We claim that $\widetilde{\Sigma}$ is irreducible in this case and then $(*)$ will follow immediately.

By (5.21) and (5.23), $\widetilde{\Sigma}$ is given by

$$(5.24) \qquad \begin{cases} y_2(y_2 + x_2^2 + \lambda_1 z_1^b) = \lambda_0 \\ w_2(x_2, 0, z_1, 0)h_1(z_1) + h_2(z_1) = 0 \end{cases}$$

where $\lambda_1 = 0$ or 1 and $\lambda_0 = \lambda(0, 0, 0, 0) \neq 0$.

Notice that $w_2(0, 0, z_1, 0)$ is a degree η polynomial in z_1 due to the way we choose γ. So if we solve $w_2(x_2, 0, z_1, 0)h_1(z_1) + h_2(z_1) = 0$ for x_2, we obtain $x_2 \in \mathbb{C}(z_1)$ and $x_2 \notin \mathbb{C}$.

It is not hard to see that $\widetilde{\Sigma}$ is irreducible if $(x_2^2 + \lambda_1 z_1^b)^2 + \lambda_0$ is a perfect square in $\mathbb{C}(z_1)$ for some $x_2 \in \mathbb{C}(z_1)$ and x_2 nonconstant. Such x_2 does not exist by Lemma 5.3.

If $c > 4\eta\gamma$, then each of $\widetilde{\Sigma}$ and J_i have exactly two irreducible components. Let $\widetilde{\Sigma} = \widetilde{\Sigma}_1 \cup \widetilde{\Sigma}_2$ and $J_i = J_i^{(1)} \cup J_i^{(2)}$ for $i = 1, 2, ..., l$.

$\underline{c > 4\eta\gamma \text{ and } a = (2\eta - b)\gamma}$

If $b = 0$, $J_i^{(1)}$ and $J_i^{(2)}$ meet at two points for all i. If $b = 1$, $J_i^{(1)}$ and $J_i^{(2)}$ meet at a single point if and only if the corresponding $\alpha_i = 0$ and this point must be $r = (x_2 = y_2 = z_1 = t = 0)$. It is not hard to see that $\widetilde{\Sigma}_1$ and $\widetilde{\Sigma}_2$ meet at (at least) one point other than r.

Let G be the dual graph of the following components of $\widetilde{\mathcal{Y}}_0$ (let $\widetilde{\mathcal{Y}}$ be the nodal reduction of $\mathcal{Y}$)

1. $\widetilde{\Sigma}_1$ and $\widetilde{\Sigma}_2$;
2. the components dominating $J_i^{(1)}$ or $J_i^{(2)}$ for $J_i^{(1)}$ and $J_i^{(2)}$ that meet at two points;
3. the contractible components which contract to a point in $(\widetilde{\Sigma}_1 \cap \widetilde{\Sigma}_2) \cup (J_1^{(1)} \cap J_1^{(2)}) \cup (J_2^{(1)} \cap J_2^{(2)}) \cup ... \cup (J_l^{(1)} \cap J_l^{(2)}) \backslash \{r\}$.

Then Lemma 2.1 tells us that $\deg([\widetilde{\Sigma}_k]) \geq 1$ for $k = 1, 2$ and all the other vertices of G has degree at least two. Therefore, either $[\widetilde{\Sigma}_1]$ and $[\widetilde{\Sigma}_2]$ lie on the same component of G or G contains a circuit.

Let $\mu_0 = \deg h_1(z_1)$.

$\underline{c > 4\eta\gamma, \ a > (2\eta - b)\gamma \text{ and either } \mu_0 > 0 \text{ or } l > 1}$

Let $p_1 \in \widetilde{\Sigma}_1 \cap \widetilde{\Sigma}_2$. If no J_i passes through p_1, then $(*)$ follows directly from Lemma 2.1. Otherwise suppose that $p_1 \in J_1$. By (5.21) and (5.22), $\mathcal{Y}$ is locally defined by

$$(5.25) \qquad y\left(y + x^2 + t^{a-(2\eta-b)\gamma} z^{b'}\right) = \lambda t^{c-4\eta\gamma}$$

and

$$(5.26) \qquad\qquad f_3(w_2, z, t) = 0$$

at p_1, where $f_3(w_2, z, t) = f_2(w_2, z - \alpha_1, t)$, $b' = 0$ if $b = 0$ or $\alpha_1 \neq 0$ and $b' = 1$ otherwise. Notice that $f_3(w_2, z, t)$ is irreducible in $\mathbb{C}[w_2, [z]] \otimes_{\mathbb{C}} \overline{\mathbb{C}[[t]]}$ by Lemma 5.1 and $(\mu_1, b') < (\mu, b)$. Apply the induction hypothesis and we are done.

$c > 4\eta\gamma$, $a > (2\eta - b)\gamma$, $\mu_0 = 0$, $l = 1$ and $\alpha_1 \neq 0$

Notice that this case happens only if $b = 1$ due to (5.10). Following the argument of the previous case, we observe that $b' = 0$ in (5.25). So $(\mu, b') < (\mu, b)$ and the induction hypothesis still applies.

$c > 4\eta\gamma$, $a > (2\eta - b)\gamma$, $\mu_0 = 0$, $l = 1$ and $\alpha_1 = 0$

We necessarily have $\gamma = \nu(c_1(t))$ in this case. So there exists $p_1 \in \widetilde{\Sigma}_1 \cap \widetilde{\Sigma}_2$ such that $p_1 \notin J_1$. Then $(*)$ follows immediately from Lemma 2.1. $\square$

Proposition 5.10. *Let X be a flat family of analytic surfaces over disk Δ_t whose central fiber $X_0 = R_1 \cup R_2$ where $R_i \cong \Delta \times \mathbb{P}^1$ for $i = 1, 2$. Suppose that $R_1 \cap R_2 = \Delta(r)$ where $\Delta(r)$ is a disk centering at point r and $\Delta(r)$ is a closed subscheme of R_i ($i = 1, 2$). Suppose that X is locally defined by (5.16) at r when embedded to Δ_{xyzt}^4.*

Let $z \in \mathcal{O}_X$ and let $z = 0$ cut out the curve $\overline{r} = \overline{r}^{(1)} \cup \overline{r}^{(2)}$ on X_0, where $\overline{r}^{(i)} \cong \mathbb{P}^1$, $\overline{r}^{(i)} \subset R_i$ and each $\overline{r}^{(i)}$ meets $\Delta(r)$ at r with multiplicity 2, for $i = 1, 2$.

Let $\mathcal{O}_X(C)$ be a line bundle on X such that the restrictions of $\mathcal{O}_X(C)$ to $\overline{r}^{(i)}$ are $\mathcal{O}_{\mathbb{P}^1}(1)$ and let s_1 and s_2 be two global sections of $\mathcal{O}_X(C)$ which generate $H^0(\mathcal{O}_{\mathbb{P}^1}(1))$ when restricted to $\overline{r}^{(i)}$ for $i = 1, 2$.

Let $f(s_1, s_2, z, t) \in \mathbb{C}[s_1, s_2, [z, t]]$ lie in

$$f(s_1, s_2, z, t) \in \left\{ \sum_{i=0}^{\infty} \left(a_i(t)s_1 z^i + b_i(t)s_2 z^i \right) : a_i(t), b_i(t) \in \mathbb{C}[[t]] \right\}$$

and $f(s_1, s_2, z, 0) \neq 0$. Suppose that $f(s_1, s_2, z, t)$ is irreducible in $\mathbb{C}[s_1, s_2, [z]] \otimes_{\mathbb{C}} \overline{\mathbb{C}[[t]]}$.

Let $\Upsilon \subset X$ be the subscheme of X cut out by $f(s_1, s_2, z, t) = 0$. Obviously, Υ_0 consists of a multiple of $\overline{r}$ and two disks Γ_1 and Γ_2. Let μ be the multiplicity of $\overline{r}$ in Υ_0. Suppose that $\mu > 0$ and Γ_i meets $\Delta(r)$ properly if Γ_i passes through r for $i = 1, 2$.

Let $\widetilde{\Upsilon}$ be the nodal reduction of Υ. Then either Γ_1 and Γ_2 lie on the same connected component of $\widetilde{\Upsilon}_0$ or the dual graph of $\widetilde{\Upsilon}_0$ contains a circuit.

Proof. Our argument proceeds almost identically to that for Proposition 5.8.

Just as in the proof of Proposition 5.8, we can choose local coordinates (x, y, z, t) of X at r such that X is defined by (5.16) at r and the local function z is the restriction of the global function z.

We may put the defining equation in the form (5.7) and also assume $\{a_i(t)\}$ and $\{b_j(t)\}$ to satisfy (5.8) and (5.9) and to satisfy (5.10) if $b = 0$.

Let γ be the number defined by (5.11) and let $\gamma \in \mathbb{Z}$ after a base change.

The case that Γ_i passes through r has been covered by Proposition 5.9. Suppose that Γ_i does not pass through r for $i = 1, 2$. Again, we will argue by induction on $(\mu, b))$.

Let $\widetilde{X}$ be the blowup of X along the subscheme cut out by $z = t^\gamma = 0$ and let $\Upsilon' \subset \widetilde{X}$ be the proper transform of Υ. The central fiber $\widetilde{X}_0$ consists of four surfaces $R_1 \cup R_2 \cup (\overline{r}^{(1)} \times \mathbb{P}^1) \cup (\overline{r}^{(2)} \times \mathbb{P}^1)$, where $R_i \cap (\overline{r}^{(i)} \times \mathbb{P}^1) = \overline{r}^{(i)}$ $(i = 1, 2)$ and $(\overline{r}^{(1)} \times \mathbb{P}^1) \cap (\overline{r}^{(2)} \times \mathbb{P}^1) = \{r\} \times \mathbb{P}^1$.

We have the same defining equations (5.12) for Υ' and (5.14) for Υ'_0. Let $I_0 = I_0^{(1)} \cup I_0^{(2)}, I_1 = I_1^{(1)} \cup I_1^{(2)}, ..., I_l = I_l^{(1)} \cup I_l^{(2)} \subset \Upsilon'_0$ be the components of Υ'_0 defined in the same way as in the proof of Proposition 5.8. Let $p_j = I_j^{(1)} \cap I_j^{(2)}$.

Let r_k be the intersection between Γ_k and $I_0^{(k)}$ with coordinates $w = 1/z_1 = 0$ on $\overline{r}^{(k)} \times \mathbb{P}^1 \cong \mathbb{P}^1 \times \mathbb{P}^1$. By Lemma 2.1,

(∗) $I_0^{(k)}$ and Γ_k are joined by curves contracting to r_k on $\widetilde{\Upsilon}_0$ for $k = 1, 2$.

<u>$l > 0$</u>

Take a component I_j and an analytic neighborhood U of $\widetilde{X}$ around I_j. Let $Y = U \cap \Upsilon'$. Then Y and U have all the properties described in the proposition. And in the neighborhood of p_j, U is given by $y(y + x^2 + t^{a+\gamma} z^{b'}) = t^c$. where $b' = 0$ if either $b = 0$ or $\alpha_j \neq 0$ and $b' = 1$ otherwise. The central fiber Y_0 of Y contains I_j with multiplicity μ_j plus two disks Γ'_1 and Γ'_2 attached to $I_j^{(1)}$ and $I_j^{(2)}$, respectively. Obviously, Γ'_k is a piece of $I_0^{(k)}$ for $k = 1, 2$.

To apply the induction hypothesis, we have to check that $(\mu_j, b') < (\mu, b)$, which is an easy consequence of (5.10) and the way we choose the number γ. Also we need to check that Γ'_k meets $\{r\} \times \mathbb{P}^1$ properly if Γ'_k passes through p_j. This is trivially true because $\{r\} \times \mathbb{P}^1 \not\subset I_0$. So the induction hypothesis applies. Combining with (∗), we obtain the statement of the proposition.

<u>$l = 0$</u>

Obviously, $I_0^{(1)}$ and $I_0^{(2)}$ meet at some point $p \in \{r\} \times \mathbb{P}^1$. Since Γ_k does not pass through r, $p \neq r_k$ for $k = 1, 2$. Therefore, $I_0^{(1)}$ and $I_0^{(2)}$ are joined by curves contracting to p on $\widetilde{\Upsilon}_0$. Combining with (∗),

we conclude that Γ_1 and Γ_2 lie on the same connected component of $\widetilde{\Upsilon}_0$. $\qquad\square$

Proposition 5.7 follows directly from Proposition 5.10. We will leave the details to the readers.

5.3. **Proof of Theorem 5.1.** Before we proceed, we would like to raise a simple question.

Let $\pi : X \to Y$ be a proper and dominant map between two irreducible varieties X and Y. Assume that Y is normal. Let $q \in Y$ and $p \in \pi^{-1}(q)$ be a point on the fiber over q. Is it true that for any analytic neighborhood U of p, $\pi(U)$ contains a neighborhood V of q?

In general, this is not true. For example, we may take X to be the blowup of $Y = \mathbb{P}^2$ at a point p and q to be a point on the exceptional divisor. The statement is true for π finite. More generally, by using Stein factorization, we have the following.

Proposition 5.11 (Open Mapping Principle). *Let $\pi : X \to Y$ be a proper and dominant map between two irreducible varieties X and Y. Assume that Y is normal. Let $q \in Y$ and let W be a connected component of $\pi^{-1}(q)$. Then for any analytic neighborhood U of W, $\pi(U)$ contains a neighborhood V of q.*

Suppose that for a general point $[R^s] \in \widetilde{\mathcal{R}}_g$, there is a point

$$([R^s], [\Upsilon_0], w_1, w_2, ..., w_\beta, \delta_1, \delta_2, ..., \delta_\beta, \nu_1, \nu_2, ..., \nu_\beta) \in \widetilde{\mathcal{Z}}_g$$

lying on the fiber of $\widetilde{\mathcal{Z}}_g \to \widetilde{\mathcal{H}}_g$ over $[R^s]$ such that Υ_0 contains a Type III chain $\overline{r_{ij}} \cup \overline{r_{ij}q_1} \cup \overline{q_1q_2} \cup ... \cup \overline{q_{m-1}q_m}$. Let W_0 be the index set such that

$$\{w_1, w_2, ..., w_\beta\} \cap (\overline{r_{ij}} \cup \overline{r_{ij}q_1} \cup \overline{q_1q_2} \cup ... \cup \overline{q_{m-1}q_m}) = \{w_i : i \in W_0\}.$$

Theorem 5.1 is equivalent to saying that $\sum_{i \in W_0} \delta_i \geq \sum_{l=0}^{m} \mathrm{Intsc}(\Sigma_1, E; q_l)$ (let $q_0 = r_{ij}$) and if the equality holds, $\nu_i = 1$ for $i \in W_0$.

As indicated in (5.3), we will move R^s in four steps.

Step 1. $[R^s] \in \widetilde{\mathcal{R}}_g \Rightarrow [R^s] \in \widetilde{\mathcal{R}}_g^0$

When a R^s in $\widetilde{\mathcal{R}}_g$ degenerates to a general R^s in $\widetilde{\mathcal{R}}_g^0$, the points $r_{ij} = q_0, q_1, q_2, ..., q_m$ on the Type III chain "collapse" to the point $r_{1j} = r_{2j}$. At the same time,

$$(5.27) \qquad \sum_{i=0}^{m} \mathrm{Intsc}(\Sigma_1, E; q_i) = \mathrm{Intsc}(\Sigma_1, E; r_{1j})$$

where the Σ_1 on the LHS refers to the component of Υ_0 lying on the old R^s and the Σ_1 on the RHS lies on the new R^s.

Meanwhile, the limiting singularities of the old R^s which lie on the Type III chain will degenerate to the points on $\overline{r_{1j}} \cup \overline{r_{2j}}$. So for $[R^s] \in \widetilde{\mathcal{R}}_g^0$, there is a point

$$([R^s], [\Upsilon_0], w_1, w_2, ..., w_\beta, ...) \in \widetilde{\mathcal{Z}}_g$$

lying on the fiber of $\widetilde{\mathcal{Z}}_g \to \widetilde{\mathcal{H}}_g$ over $[R^s]$ such that Υ_0 contains $\overline{r_{1j}} \cup \overline{r_{2j}}$ with multiplicity μ and

$$\{w_1, w_2, ..., w_\beta\} \cap (\overline{r_{1j}} \cup \overline{r_{2j}}) = \{w_i : i \in W_0\}.$$

In the proof of Proposition 2.2, we have shown that $\Gamma_1 \cup \Gamma_2$ does not pass through $r_{1j} = r_{2j}$ on the new R^s in $\widetilde{\mathcal{R}}_g^0$. So by (5.27),

$$(5.28) \qquad 2\mu = \sum_{l=0}^{m} \mathrm{Intsc}(\Sigma_1, E; q_l).$$

Step 2. $[R^s] \in \widetilde{\mathcal{R}}_g^0 \Rightarrow [S] \in \mathcal{Y}_g$

Let us consider the fiber $(\widetilde{\mathcal{Z}}_g)_{[R^s]}$ of $\widetilde{\mathcal{Z}}_g \to \widetilde{\mathcal{H}}_g$ over a point $[R^s] \in \widetilde{\mathcal{R}}_g^0$. Let the point

$$([R^s], [\Upsilon_0], w_1, w_2, ..., w_\beta, \delta_1, \delta_2, ..., \delta_\beta, \nu_1, \nu_2, ..., \nu_\beta)$$

with the property described above lie on a connected component Z of $(\widetilde{\mathcal{Z}}_g)_{[R^s]}$. Then for any point

$$\left([R^s], [\Upsilon_0'], w_1', w_2', ..., w_\beta', \delta_1', \delta_2', ..., \delta_\beta', \nu_1', \nu_2', ..., \nu_\beta'\right)$$

lying on the same component Z, we necessarily have that

1. Υ_0' contains $\overline{r_{1j}} \cup \overline{r_{2j}}$ with multiplicity μ;
2. it follows from Proposition 5.2 that

$$\{w_1', w_2', ..., w_\beta'\} \cap (\overline{r_{1j}} \cup \overline{r_{2j}}) = \{w_i' : i \in W_0\}.$$

If we take an analytic neighborhood U of Z and project it to $\widetilde{\mathcal{H}}_g$, by Proposition 5.11, the image $\pi(U)$ will contain a neighborhood V of $[R^s]$, where π is the map $\widetilde{\mathcal{Z}}_g \to \widetilde{\mathcal{H}}_g$. By Proposition 5.3, V contains general points of $\mathcal{Y}_g$.

For a general point $[S] \in \mathcal{Y}_g \cap V$, the fiber of $U \to \pi(U)$ over $[S]$ consists of points

$$([S], [D], w_1, w_2, ..., w_\beta, ...)$$

where

1. by Proposition 5.5 and 5.6, D contains two connected components $D_1, D_2 \in |\mathcal{O}_S(F)|$ with multiplicities μ_1 and μ_2, respectively, where $\mu_1 + \mu_2 = \mu$;

2.

$$\{w_1, w_2, ..., w_\beta\} \cap (D_1 \cup D_2) = \{w_i : i \in W_0\}.$$

Basically, if we have a family of surfaces S_t approach R^s, by Proposition 5.7, the corresponding family of curves $D_t \subset S_t$ will have components in $|\mathcal{O}_S(F)|$ with total multiplicities μ degenerating to $\overline{r_{1j}} \cup \overline{r_{2j}}$. And by Proposition 5.5, there are exactly two rational curves in $D_1, D_2 \in |\mathcal{O}_S(F)|$ in the neighborhood of $\overline{r_{1j}} \cup \overline{r_{2j}}$. So D will contain D_1 and D_2 with a total multiplicity μ. Let G be the irreducible component of D in $|\mathcal{O}_S(C + k'F)|$ for some k'.

Let W_1 and W_2 be the index sets such that

$$\{w_1, w_2, ..., w_\beta\} \cap D_j = \{w_i : i \in W_j\} \text{ for } j = 1, 2.$$

Of course, $W_0 = W_1 \cup W_2$ and $W_1 \cap W_2 = \emptyset$.

Step 3. $[S] \in \mathcal{Y}_g \Rightarrow [R^s] \in \widetilde{\mathcal{R}_g^0}

If we degenerate an elliptic K3 surface S in $\mathcal{Y}_g$ to a general R^s in $\widetilde{\mathcal{R}_g^0}$ with the corresponding $D \subset S$, described as above, degenerating to $\Upsilon_0 \subset R$, by Proposition 5.4 and 5.5, we can make D_1 degenerate to $\overline{p_1 q_1} = \overline{p_1 q_1}^{(1)} \cup \overline{p_1 q_1}^{(2)}$ and make D_2 degenerate to $\overline{p_2 q_2} = \overline{p_2 q_2}^{(1)} \cup \overline{p_2 q_2}^{(2)}$. So Υ_0 will contain $\overline{p_j q_j}$ with multiplicity at least μ_j for $j = 1, 2$. Let Υ_0 contain $\overline{p_j q_j}$ with multiplicity $\mu_j + \mu_j'$. Namely, $G_t \subset S_t$ will degenerate to a curve on R containing $\overline{p_j q_j}$ with multiplicity μ_j'.

Let W_1' and W_2' be the index sets such that

$$\{w_1, w_2, ..., w_\beta\} \cap \overline{p_j q_j} = \{w_i : i \in W_j\} \cup \{w_i : i \in W_j'\} \text{ for } j = 1, 2$$

where $([R^s], [\Upsilon_0], w_1, w_2, ..., w_\beta, ...) \in \widetilde{\mathcal{Z}_g}$ is the corresponding limit when S degenerates to R^s.

More intuitively, those w_i's for $i \in W_1' \cup W_2'$ are the extra limiting singularities we get due to the extra multiplicities μ_1' and μ_2'. We have to control the extra δ-invariant

$$\sum_{i \in W_1' \cup W_2'} \delta_i.$$

By Proposition 5.6 and Lemma 5.2,

$$\sum_{i \in W_1' \cup W_2'} \delta_i = \delta(G_t, \overline{p_1 q_1}) + \delta(G_t, \overline{p_2 q_2})$$

(5.29)

$$= 2\mu_1' + \text{Intsc}(\Gamma_1, E; p_1) + \text{Intsc}(\Gamma_1, E; q_1)$$
$$+ 2\mu_2' + \text{Intsc}(\Gamma_1, E; p_2) + \text{Intsc}(\Gamma_1, E; q_2).$$

Step 4. $\widetilde{\mathcal{R}_g^0} \Rightarrow [R^s] \in [R^s] \in \widetilde{\mathcal{R}_g}

By using the similar argument to that used in Step 2, we can show that for a $[(R^s)'] \in \widetilde{\mathcal{R}}_g$ lying in a neighborhood of $[R^s] \in \widetilde{\mathcal{R}}_g^0$, there is a point

$$\left([(R^s)'], [\Upsilon_0'], w_1', w_2', ..., w_\beta', ...\right) \in \widetilde{\mathcal{Z}}_g$$

where Υ_0' contains two Type II chains C_1 and C_2 such that

1. C_1 and C_2 degenerate to $\overline{p_1 q_1}$ and $\overline{p_2 q_2}$, respectively, when $(R^s)'$ degenerates to R^s;

2.

(5.30)
$$\sum_{q \in C_j \cap E} \mathrm{Intsc}(\Sigma_1', E; q) = \mathrm{Intsc}(\Sigma_1, E; p_j) + \mathrm{Intsc}(\Sigma_1, E; q_j)$$

$$= 2(\mu_j + \mu_j') + \mathrm{Intsc}(\Gamma_1, E; p_j) + \mathrm{Intsc}(\Gamma_1, E; q_j)$$

for $j = 1, 2$, where Σ_1' is the component of Υ_0' defined as usual;

3.
$$\{w_1', w_2', ..., w_\beta'\} \cap C_j = \{w_i : i \in W_j \cup W_j'\}$$

for $j = 1, 2$.

So applying Theorem 4.1 to $\Upsilon_0' \subset (R^s)'$, we have

(5.31)
$$\sum_{i \in W_j \cup W_j'} \delta_i \geq \sum_{q \in C_j \cap E} \mathrm{Intsc}(\Sigma_1', E; q),$$

for $j = 1, 2$. Combining with (5.29) and (5.30), we have

(5.32)
$$\sum_{i \in W_1} \delta_i + \sum_{i \in W_2} \delta_i \geq 2(\mu_1 + \mu_2) = 2\mu.$$

If the equality in (5.32) holds, the equalities in (5.31) have to hold. So by Theorem 4.1, $\nu_i = 1$ for $i \in W_1 \cup W_2 \cup W_1' \cup W_2'$. Combining with (5.28), we have proved Theorem 5.1.

5.4. **Completion of the Proof of Theorem 1.1.** With Theorem 3.1, 4.1 and 5.1 in place, Theorem 1.1 is more or less obvious.

Let $C_0, C_1, ..., C_l \subset \Upsilon_0$ be all the F-chains in Υ_0 where C_0 is the Type I chain. It follows from Theorem 3.1, 4.1 and 5.1 that

$$\delta(\Upsilon_t) = \sum_{i=0}^l \delta(\Upsilon_t, C_i)$$

$$\geq \sum_{i=0}^l \sum_{q \in C_i \cap E} \mathrm{Intsc}(\Sigma_1, E; q) - 1$$

$$= \Sigma_1 \cdot E - 1 = g.$$

Obviously, $\delta(\Upsilon_t) = g$. Hence we must have

$$\delta(\Upsilon_t, C_i) = \sum_{q \in C_i \cap E} \text{Intsc}(\Sigma_1, E; q)$$

for $i = 1, 2, ..., l$. Consequently, by Theorem 4.1 and 5.1, Υ_t only has nodes as singularities in the neighborhood of C_i for $i = 1, 2, ..., l$. And by Theorem 3.1, Υ_t only has nodes as singularities in the neighborhood of C_0. Therefore, all the singularities of Υ_t are nodes.

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